CUK-H632678-20-P035467

510·5.→
·002

AMERICAN
Journal of Mathematics

20

EDITED BY

SIMON NEWCOMB

WITH THE COOPERATION OF
A. COHEN, FRANK MORLEY, CHARLOTTE A. SCOTT
AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XXII

UNIVERSITY OF CALCUTTA
LIBRARY.
THE ADVANCEMENT OF LEARNING.

[illegible]NS PRESS

UNIVERSITY OF CALCUTTA
LIBRARY.
THE ADVANCEMENT OF LEARNING

INDEX.

PAGE

BLAKE, E. M. The Ellipsograph of Proclus, ... 146

BLICHFELDT, H. F. On a certain Class of Groups of Transformation in Space of Three Dimensions, ... 113

BOLZA, OSKAR. Remarks concerning the Expansions of the Hyperelliptic Sigma-Functions, ... 101

DICKSON, LEONARD EUGENE. Canonical Form of a Linear Homogeneous Substitution in a Galois Field, ... 121

Certain Subgroups of the Betti-Mathieu Group, . . 49

Concerning the Cyclic Subgroups of the Simple Group G of all Linear Fractional Substitutions of Determinant Unity in Two Non-Homogeneous Variables with Coefficients in an Arbitrary Galois Field, . 231

GILESPIE, WILLIAM. On the Reduction of Hyperelliptic Integrals ($p = 3$) to Elliptic Integrals by Transformations of the Second and Third Degrees, . . 257

HATZIDAKIS, N. J. Displacements Depending on One, Two, ... , k Parameters in a Space of n Dimensions, ... 154

HOLGATE, THOMAS F. Note Additional to a Former Paper "On Certain Ruled Surfaces of the Fourth Order," 27

LEHMER, DERRICK NORMAN. Asymptotic Evaluation of certain Totient Sums, . 293

LING, G. H., and MILLER, G. A. Proof that there is no Simple Group whose Order lies between 1092 and 2001, ... 13

LOVETT, E. O. A Property of Lines in n-Dimensional Space, ... 226

Families of Transformations of Straight Lines into Spheres, . . 138

Note on the Differential Invariants of Goursat and Painlevé, . . 41

Supplementary Note on Projective Invariants, ... 46

Metzler, W. H. On the Excess of the Number of Combinations in a Set which have an Even Number of Inversions over those which have an Odd Number, ... 55

Miller, G. A. On the Product of Two Substitutions, ... 185

Moore, Eliakim Hastings. Concerning Klein's Group of $(n+1)!$ n-ary Collineations, ... 336

The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions, ... 279

Petrovitch, Michel. Appareil à liquide pour l'intégration graphique de certains types d'équations différentielles, ... 1

Rettger, E. W. On Lie's Theory of Continuous Groups, ... 60

Slaught, Herbert Ellsworth. The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane, ... 343

Snyder, Virgil. Lines of Curvature on Annular Surfaces having Two Spherical Directrices, ... 96

On Some Invariant Scrolls in Collineations which leave a Group of Five Points Invariant, ... 253

Stecker, Henry Freeman. Non-Euclidian Properties of Plane Cubics, . . . 31

Wilczynski, E. J. On Continuous Binary Linearoid Groups, and the Corresponding Differential Equations and Λ Functions, ... 191

UNIVERSITY OF CALCUTTA
LIBRARY.
THE ADVANCEMENT OF LEARNING

PROFESSOR THOMAS CRAIG, PH. D.

Thomas Craig, the former editor of this JOURNAL, and Professor of Pure Mathematics in the Johns Hopkins University, died May 8, 1900, in his forty-fifth year. His connection with the AMERICAN JOURNAL OF MATHEMATICS, as editor or associate editor, continued through the greater part of its existence, being severed at the end of 1898, when failing health compelled him to retire from the editorship. Craig was connected with the Johns Hopkins University from its foundation. He was attracted thither by the desire to pursue mathematical studies under the guidance of Sylvester. From the beginning he showed an extraordinary development of the faculty of acquisition, being able to master, almost without effort, the writings of any of the great geometers to which he was attracted. The productive faculty was developed more slowly.

He was naturally among the earliest Doctors of the University, and the first, or one of the first, to graduate in mathematics. His earliest publications were two small books on hydrodynamics, and a work on projections, prepared for the U. S. Coast Survey, with which he was associated for a short period after his graduation. His most elaborate separate work was a treatise on Linear Differential Equations, embodying the course of instruction on that subject which he gave to the students of the University. A work on higher geometry, on which he was engaged, but, so far as the writer is aware, on which he had made little progress, was left unfinished at the time of his death.

He was also a frequent contributor to the pages of this JOURNAL. Among the contributions worthy of especial mention were his various papers on Theta functions, in the fifth and sixth volumes, and a memoir on Linear Differential Equations whose fundamental integrals are the successive derivatives of the same function, in the eighth volume.

During his editorship he devoted himself with great energy to the interests of the JOURNAL. The principal object of at least one of his visits abroad was to interest European geometers in it. He recognized and admired the genius of Poincaré; and two elaborate memoirs by the latter, which appeared in the seventh and eighth volumes, were believed to have been sent to the JOURNAL on Craig's personal solicitation.

As an expounder of mathematical subjects to advanced students, Craig's abilities were of a high order. His lectures were well prepared, and he spoke with rapidity, clearness and force. It may well be that only the best students were able to keep up with him, but these profited in a high degree from his expositions and entertained a permanent appreciation of his efforts for their development. Concentrating his interests almost entirely on his family and his students, rarely taking a long rest, he mingled little with men, especially in his later years, when his activities were greatly restricted by failing health.

SIMON NEWCOMB.

The writer is indebted to Dr. L. P. Eisenhart for part of the material on which this notice is based.

Appareil a liquide pour l'intégration graphique de certains types d'équations différentielles.

Par M. Michel Petrovitch à *Belgrade* (*Serbie*).

Dans mon Article, "Sur l'intégration hydraulique des équations différentielles" (American Journ. of Mathem., vol. XX, No. 4), j'ai indiqué une methode pouvant servir à intégrer graphiquement des équations différentielles du premier ordre. On peut faire varier la disposition de l'appareil et par cela même les types d'équations que la methode permet d'intégrer.

J'ai fait construire un appareil de ce genre, particulièrement facile à réaliser, pouvant servir à enregistrer directement les intégrales de certains types d'équations. Ces équations sont, il est vrai, intégrables analytiquement, mais il est commode pour les applications, d'avoir une methode rapide et sûre pour la construction mécanique de leurs courbes intégrales.

Déscription et fonctionnement de l'appareil.

Supposons que sur une plaque d'epaisseur uniforme (p. ex. sur une planche en bois) on marque un système d'axes rectangulaires *oz*, *ot* (fig. 1); qu'on y trace ensuite une courbe donnée

$$t = F(z)$$

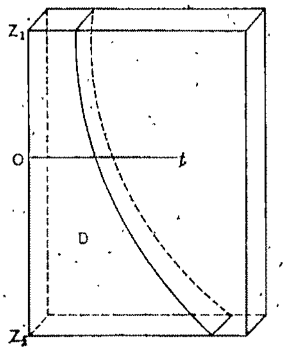


Fig. 1.

et qu'on découpe la surface limitée par deux droites z_1 et z_2, la courbe ainsi tracée et l'axe des z (si la courbe s'étend de deux cotés de oz, on découperait la plaque suivant cette courbe et les deux droites z_1 et z_2). Cette partie decoupée sera appelée dans la suite : *plaque D*.

Traçons ensuite sur une autre plaque, d'epaisseur également uniforme, un systeme d'axes ov, ow et une courbe donnée

$$w = \phi(v)$$

et comme précédemment découpons la partie de la plaque comprise entre la courbe et les deux droites v_1, v_2. Cette partie découpée sera appelée: *plaque E*.

Supposons maintenant que sur un cylindre C on ait tracé deux axes $o\xi$ et ox, de sorte que $o\xi$ soit parallèle à la base du cylindre et que ox soit une des ses génératrices. Traçons y ensuite une courbe donnée

$$x = \theta(\xi).$$

Ceci étant, supposons un appareil construit d'après le schema suivant.

Une cuve K (fig. 2), ayant la forme d'un parallelipipède, communique par

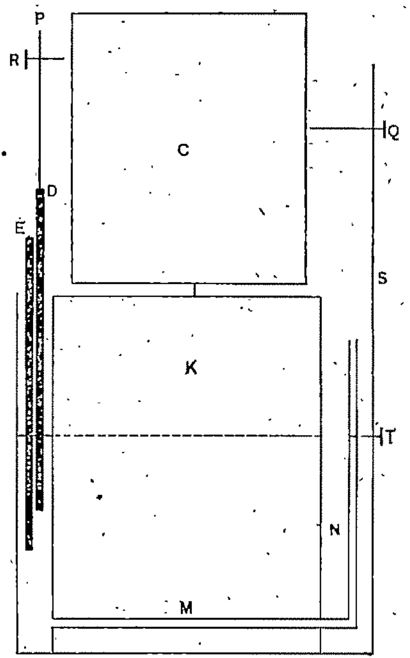


Fig. 2.

un tube horizontal M avec un autre tube vertical N. Le cylindre C est placé sur un support (p. ex. sur une caisse en bois) et peut tourner lentement autour d'un axe vertical soit par l'action d'un mécanisme d'horlogerie, soit par une manivelle qu'on tourne à la main.

On fixe la plaque E de sorte qu'elle immerge dans la cuve K et qu'elle ne puisse pas se déplacer pendant l'expérience.

La plaque D, également immergée dans la cuve, est fixée au bout d'une tige P pouvant glisser verticalement. Sur la tige P se trouve fixé un repère R; en le tenant par la main, on pourra, en le déplaçant verticalement ensemble avec la tige (et par suite ensemble avec la plaque D) faire de sorte que lorsque le cylindre C tourne, le repère se trouve constamment sur la courbe $x = \theta(\xi)$ tracée d'avance sur le cylindre.

Versons ensuite d'un liquide quelconque dans la cuve K, jusqu'à une hauteur arbitraire. Lorsque le cylindre C commence à tourner et qu'en déplaçant à chaque instant le repère R on fait de sorte qu'il se trouve constamment sur la courbe $x = \theta(\xi)$, le niveau du liquide s'elèvera ou baissera avec la longueur ξ parcourue par un point du cylindre suivant une loi dépendant des formes

1° de la courbe $x = \theta(\xi)$ tracée sur le cylindre;
2° de la courbe $t = F(z)$, suivant laquelle a été découpée la plaque D;
3° de la courbe $w = \phi(v)$, suivante laquelle a été découpée la plaque E.

Supposons qu'une tige verticale S soit placée à coté du tube N, pouvant glisser verticalement et sur laquelle se trouve fixe un repère T en bas et un crayon Q à la hauteur du cylindre. Si un observateur déplace à chaque instant le repère T de manière que sa pointe se trouve constamment à la hauteur du niveau du liquide dans le tube N et que, à l'aide d'une vis, on fixe le crayon Q sur la tige S de sorte que la distance QT soit invariable pendant ce déplacement, le crayon décrira sur le cylindre tournant une certaine courbe

$$y = \chi(\xi)$$

dépendant des trois données 1°, 2°, 3° et dont nous allons chercher l'équation différentielle.

A cet effet désignons (fig. 3) par

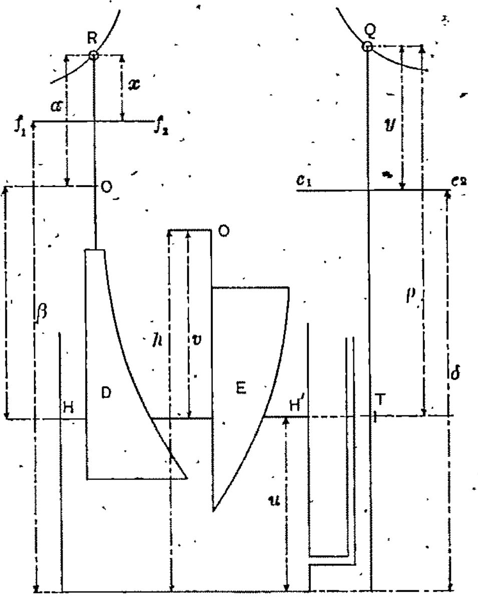


Fig. 3.

a et b les épaisseurs uniformes des plaques D et E;

l et g la largeur et l'épaisseur de la cuve K;

r le rayon du tube N;

α la distance du repère R au point O, origine de coordonnées de la courbe suivant laquelle on a découpé la plaque D;

h la distance de l'origine de coordonnées pour la plaque E au plan du fond de la cuve K;

β la distance de l'axe $f_1 f_2$ des ξ sur le cylindre C au plan du fond de la cuve K;

ρ la distance du crayon Q au niveau du liquide;

δ la distance de l'axe $e_1 e_2$ des ξ relatif à la courbe $y = \chi(\xi)$ décrite par le crayon Q, au plan du fond de la cuve. On pourrait faire coïncider cet axe avec l'axe $f_1 f_2$ relatif à la courbe décrite par le repère R. Mais comme ces deux courbes varient dans les sens inverses, il y a avantage dans la pratique, pour diminuer la hauteur du cylindre, de choisir convenablement la hauteur de cet axe;

x l'ordonnée de la courbe $x = \theta(\xi)$ tracée sur la cylindre;

y l'ordonnée de la courbe $y = \chi(\xi)$ décrite par le crayon Q;

z la distance du point O au niveau du liquide;

u la hauteur de ce niveau comptée à partir du fond de la cuve K;

v la distance de l'origine des coordonnées de la plaque E au niveau du liquide.

Les quantités a, b, l, g, r, α, h, β, ρ, δ sont des constantes pour une expérience donnée; les x, y, z, u, v sont des variables.

Ceci étant, soit HH' le niveau initial du liquide. En faisant immerger la plaque D de sorte que x se change en $x-dx$, le volume dV du liquide qui s'est élevé au-dessus du niveau primitif u sera

$$dV = gl\,du + r^2\pi\,du - b\phi(v)\,du - aF(z)\,du \tag{1}$$

et comme l'on a

$$u = y + \delta - \rho, \tag{2}$$

$$v = h - u = h + \rho - \delta - y, \tag{3}$$

en posant pour abréger

$$gl + r^2\pi - b\phi(h + \rho - \delta - y) = \Phi(y), \tag{4}$$

$$aF(z) = \Psi(z) \tag{5}$$

on aura

$$dV = [\Phi(y) - \Psi(z)]\,dy. \tag{6}$$

Mais ce volume est égal au volume du liquide déplacé par l'immersion de la plaque D. Comme la largeur de la partie immergée audessous du niveau HH' est $F(z)$, son épaisseur a et la hauteur $-dx$, son volume sera

$$-aF(z)\,dx = -\Psi(z)\,dx.$$

d'où l'équation

$$[\Phi(y) - \Psi(z)]\,dy = -\Psi(z)\,dx. \tag{7}$$

Mais on a à chaque instant

$$u + z + \alpha = x + \beta \tag{8}$$

d'où

$$\left.\begin{aligned} z &= x - y + \lambda, \\ x &= z + y - \lambda, \end{aligned}\right\} \tag{9}$$

avec

$$\lambda = \beta + \delta - \alpha - \rho; \tag{10}$$

par suite on peut écrire l'équation (7) sous l'une ou l'autre des formes suivantes:

$$\Phi(y)\,dy + \Psi(z)\,dz = 0, \tag{11}$$

$$[\Phi(y) - \Psi(x - y + \lambda)]\frac{dy}{dx} + \Psi(x - y + \lambda) = 0. \tag{12}$$

Telle est l'équation différentielle du problème. Si

$$\eta(z, y, C) = 0 \tag{13}$$

est l'intégrale générale de (11), celle de (12) sera

$$\eta(x - y + \lambda, y, C) = 0, \tag{14}$$

et inversement : si

$$f(x, y, C) = 0$$

est l'intégrale de (12), celle de (11) sera

$$f(z + y - \lambda, y, C) = 0.$$

Par suite, si la courbe, tracée sur le cylindre et le long de laquelle on fait déplacer le repère R, est

$$x = \theta(\xi),$$

et si l'intégrale générale de (11) est (13), la courbe $y = \chi(\xi)$ tracée par le crayon Q sera

$$\chi[\theta(\xi) - y + \lambda, y, C_1] = 0,$$

C_1 étant une valeur particulière de la constante d'intégration, déterminée p. ex. par la condition imposée à la courbe $y = \chi(\xi)$ de passer par un point donné $\xi = \xi_0$, $y = y_0$; ceci donne les trois équations

$$x_0 = \theta(\xi_0), \qquad z_0 = x_0 - y_0 + \lambda, \qquad \eta(z_0, y_0, C) = 0,$$

détérminant la valeur de C.

Remarquons aussi qu'étant données :

1° la courbe $x = \theta(\xi)$, tracée d'avance ;
2° la courbe $y = \chi(\xi)$ tracée sur le cylindre par le crayon Q ;
3° les valeurs initiales ξ_0, z_0, y_0 (liées d'ailleurs par

$$z_0 = \theta(\xi_0) - y_0 + \lambda$$

et dont deux, par conséquent, sont arbitraires), on déterminera la valeur de z correspondant à une valeur donnée de y (et inversement) de la manière suivante.

Si $y = y_1$ est la valeur donnée de y, la valeur correspondante de z est l'ordonnée d'un des points d'intersection de la courbe

$$y = \theta(\xi) - y_1 + \lambda \tag{15}$$

avec les droites

$$\chi(\xi) = y_1,$$

parallèles à l'axe des y.

Inversement, si z_1 est une valeur donnée de z, la valeur correspondante de y est l'ordonnée d'un des points d'intersection de la courbe

$$y = \chi(\xi)$$

avec la courbe

$$y = \theta(\xi) - z_1 + \lambda. \tag{16}$$

Les courbes (15) et (16) ne sont autres que la courbe $x = \theta(\xi)$ deplacée dans la direction de l'axe des x.

Application à quelques problèmes.

I. D'abord, l'appareil peut servir comme *intégrateur*, permettant de calculer la valeur d'une intégrale definie

$$J = \int_{z_0}^{z_1} f(z)\,dz.$$

A cet effet supposons:

1° qu'on a donné à la plaque E la forme d'un parallélipipède, de sorte qu'on ait

$$\Phi(y) = \text{const} = B;$$

2° que la courbe $x = \theta(\xi)$ soit une droite quelconque

$$x = m\xi + n;$$

3° qu'on a donné à la plaque D une forme telle, que la fonction précédente $F(z)$ soit égale à $f(z)$, c'est-à-dire qu'on ait

$$\Psi(z) = af(z).$$

L'intégrale de (11) sera alors

$$y = -\frac{a}{B}\int_{z_0}^{z} f(z)\,dz + y_0$$

et, d'autre part, la valeur y_1, correspondant à la valeur $z = z_1$, sera l'ordonnée du point d'intersection de la droite

$$y = m\xi + \lambda_1$$

(où $\lambda_1 = n + \lambda - z_1$) avec la courbe $y = \chi(\xi)$, tracée par le crayon Q. Cette valeur y_1 étant ainsi détérminée, on aura

$$J = -\frac{B}{a}(y_1 - y_0),$$

II. L'appareil peut servir aussi comme *intégraphe*. A cet effet supposons

1° qu'on a donné à la plaque D la forme d'un parallélipipède de sorte qu'on ait

$$F(z) = \text{const} = H;$$

2° que la courbe $x = \theta(\xi)$ soit une droite quelconque

$$x = m\xi + n.$$

L'équation différentielle (12) devient alors, après avoir posé

$$dx = m d\xi;$$

$$[\Phi(y) - aH]\,dy + maH\,d\xi = 0$$

d'où

$$\int_{y_0}^{y} [\Phi(y) - aH]\,dy = -maH(\xi - \xi_0).$$

Par conséquent, d'après l'équation (4), pour que l'abscisse ξ de la courbe $y = \chi(\xi)$, tracée sur le cylindre par le crayon Q, représente à chaque instant la valeur d'une intégrale donnée

$$J = \int_{y_0}^{y} f(y)\,dy,$$

il faut et il suffit qu'on ait

$$\frac{aH - \Phi(y)}{maH} = f(y),$$

c'est-à-dire que la courbe, suivant laquelle a été découpée la plaque E, soit telle qu'on ait

$$\Phi(y) = \alpha_1 f(k - y) + \alpha_2,$$

avec

$$\alpha_1 = \frac{maH}{b}, \qquad \alpha_2 = \frac{gl + r^2\pi - aH}{b}, \qquad k = h + \rho - \delta.$$

III. L'appareil s'applique à l'intégration graphique des équations

$$f(y)\,dy + \psi(z)\,dz = 0, \tag{17}$$

où f et ψ sont des fonctions positives données.

Pour cela on donnera à la plaque D une forme telle qu'on ait

$$F(z) = \frac{1}{a}\psi(z)$$

et à la plaque E une forme telle qu'on ait

$$\phi(y) = \beta_1 - \beta_2 f(k-y),$$

avec

$$\beta_1 = \frac{gl + r^2\pi}{b}, \qquad \beta_2 = \frac{1}{b}, \qquad k = h + \rho - \delta;$$

enfin, on prendra pour le courbe $x = \theta(\xi)$ une droite quelconque. D'après ce qui a été dit précédemment, en ayant la courbe $y = \chi(\xi)$ tracée par le crayon Q et les valeurs initiales ξ_0, z_0, y_0, on en deduit facilement la valeur y correspondant à une valeur donnée de z (ou inversement), y et z étant liées par l'équation différentielle précédente.

IV. L'appareil se prête aussi à l'intégration graphique des équations

$$F\left(\xi, y, \frac{dy}{d\xi}\right) = 0, \tag{18}$$

qui ne peuvent pas s'écrire directement sous la forme (17), mais dans lesquelles les variables peuvent être séparées par un changement de la variable indépendante de la forme

$$\xi = M(y+z),$$

où z est la nouvelle variable indépendante et M une fonction donnée.

En effet, en effectuant ce changement, on aura entre y et z une équation différentielle de la forme

$$f(y)\,dy + \psi(z)\,dz = 0$$

(où nous supposerons les fonctions f et ψ positives). D'autre part, en désignant par $N(\xi)$ la fonction inverse de

$$\xi = M(t),$$

on aura $\quad N(\xi) = y + z = x + \lambda = \theta(\xi) + \lambda,$

d'où $\quad \theta(\xi) = N(\xi) - \lambda.$

Par conséquent, pour que le crayon Q trace des courbes intégrales de (18), il faut et il suffit:

1° qu'on donne à la plaque D une forme telle qu'on ait

$$F(z) = \frac{1}{a}\psi(z)$$

et à la plaque E une forme telle qu'on ait

$$\phi(y) = \beta_1 - \beta_2 f(k - y),$$

avec

$$\beta_1 = \frac{gl + r^2\pi}{b}, \qquad \beta_2 = \frac{1}{b}, \qquad k = h + \rho - \delta;$$

2° que la courbe $x = \theta(\xi)$ soit la suivante:

$$x = N(\xi) - \lambda$$

où λ a la signification précédente.

V. L'appareil se prête à la résolution du problème suivant: étant donnée une fonction arbitraire

$$\tau = \Omega(t)$$

et une courbe quelconque déjà tracée $x = \theta(\xi)$, tracer la courbe

$$y = \Omega[\theta(\xi)].$$

En remarquant que si cette courbe est celle décrite par le crayon Q, on aura

$$y = \Omega(x) = \Omega(z + y - \lambda)$$

et par suite y, définie comme fonction de z par cette équation, doit satisfaire à l'équation précédente

$$\Phi(y)\,dy + \Psi(z)\,dz = 0$$

(cette condition étant d'ailleurs suffisante). Cette condition ne dépend pas de la forme de la courbe $x = \theta(\xi)$ et une fois qu'on l'a réalisée pour une quelconque des ces courbes, elle sera satisfaite pour une courbe quelconque.

On la réalisera en donnant aux plaques D et E des formes convenables. Remarquons que la dernière condition revient à celle-ci: si

$$t = \Delta(\tau)$$

est la fonction inverse de

$$\tau = \Omega(t),$$

z définie par

$$z = \Delta(y) - y + \lambda,$$

doit satisfaire à l'équation (11), ce qu'on peut réaliser de bien de façons. P. ex. on donnera à la plaque D la forme d'un parallélipipède et à la plaque E une forme telle qu'on ait

$$1 - \Delta'(y) = \Phi(y),$$

ou bien, en donnant à E la forme d'un parallélipipède de largeur B et à D une forme telle qu'on ait

$$F(z) = -\frac{B}{a}\chi'(z).$$

où l'équation $y = \chi(z)$ représente la solution de

$$y = \Omega(z + y - \lambda)$$

par rapport à y etc.

Au lieu d'avoir une plaque mobile et une fixe, on peut en avoir plusieurs. Désignons les plaques mobiles par D_i et les plaques fixes par E_i. Supposons que toutes les plaques D_i se meuvent indépendemment les unes des autres et qu'à la plaque D_i correspond la courbe

$$x = \theta_i(\xi),$$

tracée sur le cylindre. En désignant comme précédemment par

$$t_i = F_i(z),$$

la courbe suivant laquelle on ait découpé la plaque D_i et par a_i l'épaisseur de celle-ci; par

$$w_i = \phi_i(v_i),$$

la courbe suivant laquelle on a découpé E_i et par b_i l'épaisseur de celle-ci, on trouve facilement que la variation du niveau du liquide, contenu dans la cuve K, lorsque le cylindre se met en mouvement, sera réglée par l'équation différentielle

$$\Phi(y)\,dy + \sum \Psi_i(z_i)\,dz_i = 0, \tag{19}$$

où l'on a posé pour abréger

$$\Phi(y) = gl + r^2\pi - \sum b_i\phi_i(k_i - y),$$
$$\Psi_i(z_i) = a_i F_i(z_i),$$

les k_i étant certaines constantes. En désignant par

$$\varpi(y, z_1, z_2, \ldots, z_k) = 0, \tag{20}$$

la relation entre y et les z_i, exprimée par l'équation

$$\int \Phi(y)\,dy + \sum \int \Psi_i(z_i)\,dz_i + \text{const.}, \tag{21}$$

la courbe décrite par le crayon Q est celle, dont l'équation

$$y = \chi(\xi),$$

s'obtient en éliminant $z_1, z_2, \ldots, z_k$ entre (20) et les k équations

$$\left.\begin{array}{l} z_1 = \theta_1(\xi) - y + \lambda_1, \\ z_2 = \theta_2(\xi) - y + \lambda_2, \\ \dots\dots\dots\dots\dots\dots \\ z_k = \theta_k(\xi) - y + \lambda_k, \end{array}\right\} \tag{22}$$

où λ_i sont certaines constantes analogues à la constante λ précédente.

On peut ainsi, en partant d'une équation quelconque de la forme (19) et en y remplaçant les z_i par leurs valeurs (22) (où les fonctions θ_i et les constants λ^i peuvent être quelconques), former des équations du premier ordre, en ξ et y, dont les courbes intégrales peuvent être enregistrées par le crayon Q de l'appareil.

Remarque. Je saisis l'occasion de signaler quelques erreurs de calcul, qui se sont glissé dans mon Article cité; on retablira aisément les formules exactes en remplaçant dans cet Article l'équation (1) par la suivante:

$$[\Phi(y) - F(z)]\, dy = -F(z)\, dx$$

(comme il est facile de s'en rendre compte) et par suite en modifiant les formules qui en resultent. Cela, d'ailleurs, n'atteint pas la signification de la méthode et ne change que des types intégrables par les procédés décrits.

Proof that there is no Simple Group whose Order lies between 1092 and 2001.

By G. H. Ling and G. A. Miller.

In this paper it is proposed to continue the search for simple groups of low orders. This work was begun by Hölder,[*] who examined all groups whose orders do not exceed 200. It was carried by Cole[†] up to the order 660 and by Burnside[‡] up to the order 1092. It is here proven that no simple groups exist having orders between 1092 and 2001.

The work is, in general, based on the theorems of Sylow. For the sake of clearness it may be desirable to state the principal theorems used. In the first place, Burnside shows that all groups of odd order within the limits given are composite, and he proves also that a group of even order cannot be simple unless its order is divisible by 12, 16, or 56.[§] We have also the theorems:

1. *The only simple groups, whose orders are the product of four or of five primes, are groups of order* 60, 168, 660 *and* 1092; *and no group whose order contains less than four prime factors is simple.*

2. *If H is one of n conjugate subgroups of a group G of order N, and if N is not a factor of $n!$, G cannot be simple.*[||]

3. *Groups of order* $p_1^{\alpha}p_2$, $p_1^{\alpha}p_2^{2}$, *and* $p_1^{\beta}p_2^{\alpha}$ $(\beta = 1, 2 \ldots . 5;\ p_1 < p_2)$ *are soluble.*[**]

After rejecting all the orders not divisible by 12, 16, or 56, we have left 118 possible orders. We are enabled to reject the following ones:

By theorem 1: 1116, 1128, 1136, 1140, 1164, 1168, 1212, 1236, 1264,

---

[*] Math. Annalen, vol. XL, pp. 55–88.

[†] American Journal, vols. XIV and XV.

[‡] Proceedings of London Mathematical Society, vol. XXVI (1895), pp. 333–339.

[§] Burnside, "The Theory of Groups." Arts. 256, 260.

[||] Hölder (l. c.).

[**] Frobenius, Berliner Sitzungsberichte, 1895, p. 190, and Burnside (l. c.).

1272, 1284, 1288, 1308, 1328, 1332, 1356, 1380, 1416, 1424, 1428, 1452, 1464, 1476, 1524, 1548, 1552, 1572, 1596, 1608, 1616, 1624, 1644, 1648, 1668, 1692, 1704, 1712, 1716, 1736, 1740, 1744, 1752, 1788, 1808, 1812, 1860, 1884, 1896, 1908, 1932, 1956, 1992.

By theorem 2: 1176, 1456, 1500, 1960.

By theorem 3: 1152, 1184, 1216, 1280, 1296, 1312, 1376, 1408, 1472, 1504, 1536, 1568, 1600, 1664, 1696, 1792, 1856, 1888, 1936, 1944, 1952, 1984, 2000.

By Sylow's theorem: 1248, 1360, 1368, 1392, 1488, 1632, 1760, 1776, 1840, 1904, 1968.

There remain 28 possible orders which require special treatment, viz.

$1104 = 2^4.3.23$, $1120 = 2^5.5.7$, $1188 = 2^2.3^3.11$, $1200 = 2^4.3.5^2$,
$1224 = 2^3.3^2.17$, $1232 = 2^4.7.11$, $1260 = 2^2.3^2.5.7$, $1320 = 2^3.3.5.11$,
$1344 = 2^6.3.7$, $1400 = 2^3.5^2.7$, $1404 = 2^2.3^3.13$, $1440 = 2^5.3^2.5$,
$1512 = 2^3.3^3.7$, $1520 = 2^4.5.19$, $1560 = 2^3.3.5.13$, $1584 = 2^4.3^2.11$,
$1620 = 2^2.3^4.5$, $1656 = 2^3.3^2.23$, $1680 = 2^4.3.5.7$, $1728 = 2^6.3^3$,
$1764 = 2^2.3^2.7^2$, $1800 = 2^3.3^2.5^2$, $1824 = 2^5.3.19$, $1836 = 2^2.3^3.17$,
$1848 = 2^3.3.7.11$, $1872 = 2^4.3^2.13$, $1920 = 2^7.3.5$, $1980 = 2^2.3^2.5.11$.

We now proceed to the consideration of these cases.

$$1104 = 2^4.3.23.$$

A group of order 1104 must contain 1 or 24 conjugate subgroups of order 23. If it were simple, it would, therefore, be represented as a doubly-transitive group of degree 24. The maximal group of degree 23 would be of order 46. As this group of order 46 would contain negative substitutions, a group of order 1104 cannot be simple.

$$1120 = 2^5.5.7.$$

A group of order 1120 must contain 1 or 8 subgroups of order 7. A simple group of this order could, therefore, be represented as a doubly-transitive group of degree 8. Since there is no transitive group of degree 7 and order 140,* this is impossible.

$$1188 = 2^2.3^3.11.$$

---

* Mathieu, Comptes Rendus, vol. XLVI, p. 1048.

If simple, this group has 12 conjugate subgroups of order 11. It could then be represented as doubly-transitive in 12 elements. But there is no such group of this order.*

$$1200 = 2^4 . 3 . 5^2.$$

A simple group of order 1200 could be represented as a transitive group of degree 16, since every group of order 1200 must contain 1, 6 or 16 subgroups of order 25. This group could not be imprimitive, since there is no group of degree 8 and order 1200.† It could not be primitive since there is no primitive group of degree 16 and order 1200.‡

$$1224 = 2^3 . 3^2 . 17.$$

Since every group of this order contains 1 or 18 subgroups of order 17, a simple group of this order could be represented as a doubly-transitive group of degree 18. The maximal subgroup of degree 17 would be of order 68. The group of order 1224 would, therefore, contain a subgroup of order 8 that contains a substitution formed by four cycles of order 4 and substitutions of degree 18. Since the latter would have to be negative, the simple group is impossible. It might appear that the group of order 8 could contain 5 substitutions of order 2 and degree 16, but this is impossible, since there would be just 17.9 conjugate substitutions of this kind.

$$1232 = 2^4 . 7 . 11.$$

Every group of order 1232 must contain 1 or 56 conjugate subgroups of order 11. A simple group of order 1232 would then contain 560 operators of order 11. Hence, it could not contain 176 conjugate subgroups of order 7. It would then have to contain 22 conjugate subgroups of order 7, and could be represented as a transitive group of degree 22. There would be subgroups of order 56 and degree 21. Such a subgroup must contain 28 operators which transform the operator of order 7 into itself. This group would contain operators of order 14 and degree 21. But these operators would be negative. Hence, no simple group of this order exists.

---

* Miller, Quar. Jour. Math., vol. XXVIII, pp. 193–231.

† Mathieu (l. c.), p. 1208.

‡ Miller, Amer. Jour. Math., vol. XX, p. 229.

$$1260 = 2^2 . 3^2 . 5 . 7 .$$

A group of order 1260 has 1, 15, or 36 conjugate subgroups of order 7. If there are 36 such groups, each of them is self-conjugate in a subgroup of order 35, which is necessarily cyclical and whose operators are circular in 35 elements if the group is represented as a transitive group of degree 36. All the operators then affect 35 or 36 elements. There is no group of this order and type.*

The group, if simple, would, therefore, have 15 conjugate subgroups of order 7, and could be represented as a primitive group of degree 15. But there is no such simple group.†

$$1320 = 2^3 . 3 . 5 . 11 .$$

A simple group of this order would have 12 conjugate subgroups of order 11 and could be represented as a doubly-transitive group of degree 12. There is only one doubly-transitive group of this order and degree, and it is composite.‡

$$1344 = 2^6 . 3 . 7 .$$

A simple group of this order would have 21 conjugate subgroups of order 64. The groups of order 64 cannot be distinct. If the groups are not distinct we get, by means of the equation,§

$$1 + 2x_1 + 4x_2 + 8x_3 + 16x_4 = 21 ,$$

either (a) $x_1 \neq 0$ or (b) $x_1 = 0 , x_2 \neq 0$.

(a). Let $x_1 \neq 0$. There are then several subgroups of order 64 which have in common a subgroup of order 32. The latter is then self-conjugate in a group of order $64 (1 + 2k)$ which has 3 or 7 conjugates. The main group could then be represented as transitive in 7 elements, in which case it could not be simple.‖

(b). Let $x_1 = 0$ and $x_2 \neq 0$. Then several groups of order 64 have in common a subgroup of order 16 which is then self-conjugate in a subgroup of order $32 (1 + 2k)$. The latter group then has $\frac{64 . 21}{32 (1 + 2k)}$ conjugates. The only case that need be dealt with is that in which $k = 1$. The main group can then be represented as transitive in 14 elements. There are three transitive groups of order 1344 and degree 14, but all are composite.* There is then no simple group of this order.

---

* Jordan, Liouville's Journal, 2ième ser., vol. XVII, pp. 351–367.

† Miller, Proc. London Math. Soc., vol. XXVIII, pp. 533–545.

‡ Miller, Quar. Journal, vol. XXVIII, p. 193.

§ Burnside, Proc. London Math. Soc., vol. XXVI, p. 336.

‖ Mathieu (l. c.), p. 1208.

$$1400 = 2^3 . 5^2 . 7.$$

A simple group of order 1400 would have 56 conjugate subgroups of order 25 and 8 or 50 conjugate subgroups of order 7. If the subgroups of order 25 should be all distinct, it is clear that there could be at most 8 operators of even order and, if the group exists, a self-conjugate subgroup of order 8.

If the subgroups of order 25 should not be distinct, there would be a subgroup of order 5 self-conjugate in the whole group. There is then no simple group of this order.

$$1404 = 2^2 . 3^3 . 13.$$

A simple group of this order would have either 13 or 52 conjugate subgroups of order 27. The group could not be represented as a transitive group of degree 13.† Hence, there must be 52 conjugate subgroups of order 27, and the group could be represented as primitive in 52 elements. Hence, it would have a subgroup of order 27 and degree 51. This group would have at least one transitive constituent of degree 3. There must then be a subgroup of order 9 which would be common to several of the groups of order 27, and hence self-conjugate in the group generated by them. This would require that the groups of order 27 should not be maximal. Consequently there can be no simple group of order 1404.

$$1440 = 2^5 . 3^2 . 5.$$

If a group of order 1440 were simple, it would contain either 96 or 36 conjugate subgroups of order 5.

If it would contain 96 such subgroups, it could be expressed as a transitive group of degree 96. If it were primitive, then it would have a subgroup of degree 95 and order 15, which would be cyclical. Consequently all the transitive constituents of this latter group would be of degree 15. This is impossible, since $95 \div 15 \neq$ integer. If the group were imprimitive in 96 elements, it could be primitive and of degree either 48, 32 or 24. A primitive group of order 1440, and of degree 48, would have 48 subgroups of order 30 and degree

---

* Miller, Quar. Jour., vol. XXIX, p. 248.

† Ibid., p. 224.

47, each of which would contain *one* subgroup of order 5. We would not then have 96 subgroups of order 5. A primitive group of order 1440 and degree 32 would have 32 subgroups of order 45 and degree 31, each of which would contain *one* subgroup of order 5, giving in all 32 subgroups of order 5. A primitive group of order 1440 and degree 24 would have 24 subgroups of order 60 and degree 23. These could contain at most 6 subgroups of order 5 and degree not greater than 20. Hence, the total number would be at most $\frac{6 \cdot 24}{4} = 36$. There cannot then be 96 subgroups of order 5.

If the group has 36 conjugate subgroups of order 5, it can be represented as a primitive group of degree either 18 or 36. In the first case the group would contain 18 subgroups of order 80 and degree 17. These must contain 16 subgroups of order 5, and, consequently, a self-conjugate subgroup of order 16 and degree <17. These subgroups of order 80 could not then be maximal. The group then must be primitive in 36 elements. There must be 36 subgroups of order 40 and degree 35, each containing a substitution of degree 35 and order 5, and at least one of its transitive constituents must be of degree 5. The order of this constituent could not be 5, for the order of each constituent must be divisible by the same prime numbers. If the order were 20, the group of order 40 and degree 35 would contain a substitution of order 2 and degree 20 that would be commutative to each one of its substitutions. Hence, this group of order 40 and degree 35 would contain at least $35 - 20 = 15$ substitutions of degree 20 and order 2. This is impossible, since only the 10 which correspond to substitutions of order 2 in the constituent group of degree 5 and order 20 could be of order 2.

It remains to consider the case in which the order of the constituent of degree 5 is 10. The group of order 4 which corresponds to identity of this constituent has at least one substitution of order 2 and degree 20 which is commutative to every substitution of the group of order 40 and degree 35. Hence, the latter group must contain at least 15 other substitutions of degree 20 and order 2. We shall divide the problem into three parts: (1) when the subgroup of order 4 is cyclical, (2) when it is non-cyclical and of degree 20, (3) when it is non-cyclical and of degree 30.

(1). In this case the substitutions of order 4 must contain 5 cycles of order 4 and 5 of order 2, and there must be a transitive constituent of degree 20 and order 40 that contains a cyclical subgroup of order and degree 20. The operators in the tail of this group must transform the operators of the head either into the 9th or into the 19th power. In the former case there would be only 10 operators of order 2 in the tail, and in the latter case the degree of each of the operators of order 2 in the tail of the group of order 40 and degree 35 would exceed 20, since the degree of the part of these operators that belongs to the transitive constituent of degree 20 could not be less than 18.

(2). In the second case the given group of order 4 and of degree 20 is found in 15 conjugates of the group of order 40 and degree 35. All the subgroups which correspond to this group of order 4 and degree 20 in these conjugates must be contained in the first group of order 40, and a substitution that belongs to the group of order 4 cannot be transformed into itself by every substitution of a conjugate to the first group of order 40. Hence, it is impossible to construct the required 15 subgroups of order 4 with the substitutions of the first group of order 40.

In the third case, any two of the substitutions of order 2 in the group of degree 30 and order 4 have 10 elements in common. If each of these substitutions of order 2 is commutative to every substitution of the subgroup of order 40, then every one of the conjugates of this group of order 40 must contain substitutions of the group of order 40 for its 4 operators that are commutative to every operator of the group. This is impossible, since no two of these groups of order 4 can contain any common operator except identity. If only one of the three operators of order 2 in the given subgroup of order 4 is commutative to all the substitutions of the group of order 40 and degree 35, this group must contain a transitive constituent of degree 20 and order 40 whose tail contains only substitutions of degree 20. Hence, this group of order 40 and degree 35 cannot contain 15 substitutions of degree 20 and order 2. This completes the proof that there is no simple group of order 1440.

$1512 = 2^3 . 3^3 . 7$.

In a simple group of this order there would be 36 subgroups of order 7. The group could, then, be represented as a transitive group of degree 36. It could not be represented as an imprimitive group of degree 36, for it could then be represented as a primitive group of degree 18, and hence would contain a group of order 84 and of degree 17 which would have only one subgroup of order 7. In this case there would not be 36 subgroups of order 7. The groups of order

1512 and degree 36 must then contain a maximal subgroup of order 42 and of degree 35. Since the order of each transitive constituent of this subgroup must involve tho same prime factors,* this group would consist of a simple isomorphism between groups of order 42. If one of these constituents should be of degree 7, the group of order 42 would be simply isomorphic to the group of order 42 and of degree 7. If none of the constituents should be of degree 7, the constituents would be of degrees 14 and 21. There are only two transitive groups of degree 14 and of order 42. As one of these contains only a single substitution of order 2, it cannot be represented as a transitive group of degree 21. The group of order 42 would then have to be simply isomorphic to the metacyclic group of degree 7. This latter group can be represented as a transitive group of each of the degrees 7, 14, 21 and 42. In each case it contains negative substitutions. Hence, the group of order 42 and of degree 35 would have to contain an even number of transitive constituents, viz. 14, 7, 7, 7 or 14, 21. In the former case, the main group would be of class 30, and in the latter, of class 32. In the former case the substitutions of order 2 and 6 would be of degree 32; in the latter case, those of order 3 would be of degree 33.

If the group of order 42 and of degree 35 should have the constituents 14, 7, 7, 7, it would contain $\frac{7 \cdot 36}{4} = 63$ subgroups of order 6, each of which would be generated by a substitution containing 5 cycles of order 6 and one transposition. The group of order 24 which would transform such a subgroup into itself, would therefore contain a transitive constituent of degree 2. As the group of order 42 could not contain a subgroup of order 12, this is impossible.

It remains to consider the case in which the group of order 42 and of degree 35 would involve the two constituents of degrees 14 and 21. The group of order 24 which would transform one of its substitutions of order 2 into itself would have to contain 4 cyclical subgroups of order 6 and of degree 35. Hence, the subgroup of order 8 contained in the whole group would have to be transformed into itself by at least 24 substitutions. If this subgroup of order 8 should be transformed into itself by just 24 substitutions, there would be 63 such subgroups, and the main group could be represented as a transitive group of degree 63. This could not be primitive since the group of order 8 in the group of degree 62 and of order 24 would have to contain a constituent of degree 2.

---

* Jordan, "Traité des Substitutions," p. 284.

If it were imprimitive, it would be simply isomorphic to a primitive group of degree 21. The group of order 72 and degree 20 would then contain 3 subgroups of order 8 and of degree 20. These would be transformed according to the symmetric group of degree 3, and hence the group of order 72 would have to contain a self-conjugate subgroup of order 4 and of degree 20. The group of order 72 would contain 4 subgroups of order 9, which it would transform according to the alternating or symmetric group of degree 4, and hence it would contain also a self-conjugate subgroup of order 3 and of degree 12. It would, therefore, have to contain a constituent of degree 12, which would be either transitive or would contain two constituents of degree 6. The substitutions of order 3 in the constituent of degree 8 would have to be of degree 6. As the constituent of degree 12 could not contain $20 - 12 = 8$ substitutions of degree 6 and order 3, the whole group could not contain 63 subgroups of order 8.

If the groups should contain 21 subgroups of order 8, the degree of each of the transitive constituents contained in the self-conjugate subgroup of order 8 which occurs in the subgroup of order 72 and degree 20, would be divisible by 4. It can be proven as before that this group of order 72 would contain a self-conjugate subgroup of order 3 and degree 12. The constituent of degree 12 would be transitive and the proof given above would apply to this case.

It is impossible that the number of subgroups of order 8 should be 27 or 189, since the number of substitutions that transform the subgroup of order 8 into itself is divisible by 3.

$$1520 = 2^4 . 5 . 19.$$

A simple group of this order would have 20 conjugate subgroups of order 19. The group could be represented as doubly-transitive and of degree 20. The group of order 76 and degree 19 would have a substitution of order 38 which could not be represented by means of 19 elements. Consequently there can be no simple group of this order.

$$1560 = 2^3 . 3 . 5 . 13.$$

A simple group of this order would have 40 conjugate subgroups of order 13, and 26 or 156 conjugate subgroups of order 5. The group could then be represented as transitive and of degree 40. If the subgroups of order and degree

39 were transitive, the operators of the group would all be of degree 39 or 40. No such group of this order can exist.[*] If the subgroups of order 39 were intransitive, each of them would contain 26 operators of degree 36, forming 13 conjugate subgroups of order 3 and degree 36. The whole group would then contain $\frac{40.13}{4} = 130$ conjugate subgroups of order 3. The subgroups of order 3 are self-conjugate in groups of order 12 which would contain operators of order 6. From the consideration of the representation of the group as transitive and of degree 130, it becomes clear that there would be 129 conjugate subgroups of order 6. Since the groups of order 5 are not commutative to the groups of order 3, it follows that the number of groups of order 5 must be 126. The group would then contain 480 operators of order 13, 624 of order 5, and 520 of order 6 or 3. But this is impossible. Hence there is no simple group of this order.

$$1584 = 2^4 . 3^2 . 11.$$

A simple group of this order would have 12 or 144 conjugate subgroups of order 11. The group cannot be transitive and of degree 12. Hence, it must be transitive and of degree 144. In this case all of the operations would affect 144 or 143 elements. Such a group cannot exist.[†]

$$1620 = 2^2 . 3^4 . 5.$$

A simple group of this order would have 10 conjugate subgroups of order 81. But there is no transitive group of order 1620 and degree 10.[‡] Hence, there is no simple group of this order.

$$1656 = 2^3 . 3^2 . 23.$$

There would be 24 conjugate subgroups of order 23 in a simple group of this order. The group could then be represented as doubly-transitive and of degree 24. The subgroup of order 69 and degree 23 would be cyclical. But no such subgroup exists. There is then no simple group of this order.

---

[*] Jordan, Liouville's Jour., 2ième ser., vol. XVII, pp. 351–367.

[†] Jordan (l. c.).

[‡] Cole, Quar. Jour., vol. XXVII, p. 39.

$1680 = 2^4 . 3 . 5 . 7$.

The number of subgroups of order 5 in a simple group of order 1680 would be 16, 21, 56, or 336. There would also be 120 subgroups of order 7, so that there could not be 336 of order 5. There could not be 16 subgroups of order 5, for there is no simple group of this order, primitive and of degree 8 or 16. The number of groups could not be 56, for the group would then permute them primitively or imprimitively according to a group of degree 56. If primitively, there would be a subgroup of order 30 and of degree 55 which would contain a cyclical group of order 15 and whose systems of intransitivity would consequently be multiples of 15. But $55 \div 15 \neq$ integer. Hence, this case could not occur. If imprimitively, the only case that requires consideration is that in which the systems of imprimitivity contain 2 elements. There would be operations leaving some system unchanged, and such an operation, if it left one of the elements unchanged, would have to leave the other unchanged. But a subgroup of order 5 leaves itself unchanged and does not leave unchanged any other group of order 5. The number of subgroups of order 5 could not, therefore, be 56. If the number were 21, there would be a subgroup of order 80 and of degree 20 in which the group of order 5 would be contained self-conjugately. In this group 20 operators would be commutative to the operations of order 5, and would form a regular group. The group of order 80 and degree 20 would then be transitive, and the average degree of this operation would be 19. The average degree of the subgroup of order 20 is 19. Hence, the average degree of the remaining 60 operators would have to be 19. If each were of degree 19, this would give rise to 630 distinct operations. In addition, there would be 399 from the groups of order 20 and 720 from the groups of order 7, or in all more than 1680. It can be shown that, in case all of the tail of 60 in the group of order 80 were not of degree 19, the total number would be still higher Hence, the group could not have 21 subgroups of order 5 and no simple group of the order could exist.

$1728 = 2^6 . 3^3$.

A group of this order has 1, 9, or 27 conjugate subgroups of order 64. The group, if simple, could not be transitive and of degree 9.[*] Hence, it could be represented as primitive and of degree 27. It has then a subgroup of order 64 and degree 26, one of whose transitive constituents is of degree 2. Hence, there is a group of order 32 common to several subgroups of order 64 which are then not maximal. Hence, there is no simple group of this order.

---

[*] Cole, Quar. Jour., vol. XXVI, p. 386.

$$1764 = 2^2 . 3^2 . 7^2.$$

The order of this group is of the form $p^2 q^2 r^2$. Consequently the group is composite.[*]

$$1800 = 2^3 . 3^2 . 5^2.$$

A simple group of this order would contain 36 subgroups of order 25 and could, therefore, be represented as a transitive group of degree 36. The subgroup of degree 35 and order 50 could not be maximal because its subgroup of order 25 would contain a transitive constituent of degree 5. Hence, the group, if simple, must be primitive in 18 elements. In this case the subgroup of degree 17 would be of order 100, and hence the group could not contain 36 subgroups of order 25.

$$1824 = 2^5 . 3 . 19.$$

A group of this order has 1 or 96 conjugate subgroups of order 19. If it had 96 such subgroups, it could be represented as transitive and of degree 96. It would have subgroups of degree 95 and order 19, all of whose operations would affect 95 elements. All of the operations of the group would then affect 96 or 95 elements. Such a group cannot exist.[†] Hence, a group of this order must be composite.

$$1836 = 2^2 . 3^3 . 17.$$

A simple group of order 1836 would have 18 conjugate subgroups of order 17, and hence could be represented as doubly-transitive and of degree 18. There would then be a subgroup of degree 17 and order 102. But no such group exists.[‡] Hence, there is no simple group of this order.

---

[*] Maillet, Quar. Jour., vol. XXIX, p. 250.

[†] Jordan (l. c.).

[‡] Miller, Quar. Jour., vol. XXXI, pp. 49–57.

$1848 = 2^3 . 3 . 7 . 11.$

A simple group of order 1848 would have 12 or 56 conjugate subgroups of order 11, and 22 conjugate subgroups of order 7. There is no group of this order which is transitive and of degree 12. There must then be 56 conjugate subgroups of order 11, each self-conjugate in a subgroup of order 33. Representing the group as transitive and of degree 22, the operator of order 11 would have 2 cycles and the group of order 33 could not be constructed. Then no simple group of this order can exist.

$1872 = 2^4 . 3^2 . 13.$

Any group of this order has 1 or 144 conjugate subgroups of order 13. If there were 144 such subgroups, the group could be represented as transitive and of degree 144. All of the operations of the group would then affect either 143 or 144 elements. Such a group of this order cannot exist.* Hence, all groups of this order are composite.

$1920 = 2^7 . 3 . 5.$

A simple group of this order would have 15 conjugate subgroups of order 128. But there is no simple primitive group of this order and of degree 15.† Hence, there is no simple group of this order.

$1980 = 2^2 . 3^2 . 5 . 11.$

A group of this order, if simple, would have 12 or 45 conjugate subgroups of order 11. There is no doubly-transitive group of order 1980 and degree 12. Hence, there would be 45 conjugate subgroups of order 11. The group could be represented as transitive and of degree 45. The group of order and degree 44, which leaves one element unaffected, could not be transitive; for then all of the operators of the whole group would be of degree 44 or 45. Such a group of this order cannot exist.‡ If the group of order and degree 44 is intransitive, the operations of order 11 are transformed into themselves by 22 substitutions. Hence, the group contains a cyclical subgroup of order 22 and degree 44. The tail of this group contains 11 operations of order 2 and degree 44, and 11 operations of order 2 and degree 40. There are then $\frac{11 \times 45}{5} = 99$ conjugate groups of order 2 and degree 40. Representing the group as transitive and of degree 99, the subgroup leaving one element unaffected is of order 20, and contains *one* subgroup of order 5 which leaves unaffected $4 + 5k$ elements, and consequently has $\frac{99}{4 + 5k}$ conjugates. It is easy to show, however, that the number of subgroups of order 5 must be either 36 or 66. Hence, there can be no simple group of order 1980.

---

* Jordan (l. c.).

† Miller, Proc. London Math. Soc., vol. XXVIII.

‡ Jordan (l. c.).

ITHACA, *July*, 1899.

Note Additional to a Former Paper "On Certain Ruled Surfaces of the Fourth Order."

BY THOMAS F. HOLGATE.

In a paper entitled "On Certain Ruled Surfaces of the Fourth Order," published in volume XV of this Journal, I discussed those surfaces of the fourth order which may be generated by two projectively related sheaves of planes of the second order, or by the reciprocal method of two projectively related conics. It was there shown that when the collinear bundles in which the two projectively related sheaves of planes are chosen have a self-corresponding ray, the surface generated will have as nodal curve two straight lines which may be either real and distinct, coincident, or imaginary, and a double generator, namely, the self-corresponding ray of the bundles. The surface for which the nodal lines are real and distinct, F_5^4, and that for which the nodal lines are coincident, F_6^4, were discussed, but no mention was made of the surface for which the nodal lines are imaginary, though the existence of such a surface must have been in mind at the time.* From the algebraic point of view it would perhaps be unnecessary to distinguish between these three surfaces, but from the geometrical standpoint a study of the separate surfaces is of considerable interest.

Before taking up the study of this surface, however, I wish to call attention to a misstatement in my former paper.

In article 40, page 380, the double generator on the surface F_5^4 is spoken of as being *real* or *imaginary* according as the points O_1 and O_2 in which the double generator meets the nodal lines k_1 and k_2 respectively, lie without or within the generating cones. It should have been stated that the double generator *lies actually on the surface* or *is isolated* according as these points lie without or within the generating cones. In the discussion of the subforms of this surface in article 41, again in article 45, and in the treatment of the surface F_6^4 (art. 47), the same mistake is made and should be similarly corrected.

---

*See former paper, article 39, page 379.

Two collinear bundles of planes S and S', which have a self-corresponding ray SS', generate a line congruence of the first order and first class. When the two self-corresponding planes of the bundles, α_1 and α_2, are real and distinct, the singular points of the congruence lie upon two real and distinct straight lines k_1 and k_2, which are gauche to each other and meet the self-corresponding ray SS' in points O_1 and O_2. If these planes are coincident, k_1 and k_2 coincide. If the self-corresponding planes are imaginary, there are no real singular points of the congruence outside the ray SS'.

Suppose that in the latter case two projectively related sheaves of planes of the second order Φ_7 and Φ_7', enveloping cones K_7 and K_7' respectively, be chosen. These will generate a ruled surface of the fourth order, F_7^4, whose generators are rays of the congruence that is generated by S and S'. No two generators of this surface can intersect, since the congruence in which it lies has no real singular points; consequently, the surface has no real nodal lines. Two planes of the sheaf Φ_7 and the two corresponding planes of the sheaf Φ_7' will, in general, pass through the self-corresponding ray SS'. This ray will, therefore, be a double generator of the surface.

If the cones K_7 and K_7' be so chosen that the ray SS' lies outside either of them, and, consequently, also outside the other, the double generator will lie actually upon the surface and the two sheets of the surface through it will be distinct. If SS' lies on the cones, the surface will be torsal all along this generator, while if this self-corresponding ray lies inside the cones, the double generator will be isolated.

The section of the surface F_7^4, made by an arbitrary plane, is a quartic curve having a real double point where the plane intersects the double generator and two imaginary double points. The section by a plane through an arbitrary generator consists of this generator and a cubic curve which has a double point at the intersection of the plane with the double generator and which intersects the generator of its plane in only one real point, namely, the point at which the plane is tangent to the surface. The real double point in both these cases is a crunode a cusp, or an acnode, according as the double generator of the surface is crunodal, torsal, or isolated.

The section of the surface made by a plane through the double generator consists of this line counting doubly and a conic which intersects the line in those points at which the plane is tangent to the surface. These points are real and distinct, coincident, or imaginary, according as the double generator again is crunodal, torsal, or isolated. The planes through the double generator are the only real bitangent planes of the surface.

That the surface F_7^4 is self-reciprocal may be shown by a consideration of the sections made by two bitangent planes, exactly as was done for the surface F_5^4 in the original paper.

For convenience of reference I here append a table showing the principal characteristics of the various forms of Ruled Surfaces of the Fourth Order, including those which do not yield themselves to the projective methods of these papers.

SURFACES WHICH ADMIT A TRINODAL QUARTIC SECTION. $p=0$.

SPECIES.	Character of the real nodal curve.	Forms constituted by real bitangent planes.	Reciprocal surface.	CORRESPONDING SPECIES.		
				Cayley.	Cremona.	Salmon.
F_1	A twisted cubic; no straight line director.	A sheaf of the third order.	Of the same form.	10	1	VI.
F_2	A twisted cubic, with a straight line director.	A sheaf of the first order, each plane tritangent.	F_2'	8	7	VII.
F_3	A conic and an intersecting straight line which is not a generator.	A sheaf of the first order and one of the second order.	Of the same form.	7	2	VIII.
F_4	A conic and an intersecting straight line which is also a generator.	A sheaf of the first order, each plane tritangent.	F_4'	11	4	IX.
F_5	Two distinct straight lines and a double generator.	Three distinct sheaves of the first order.	Of the same form.	2	5	X, say.
F_6	One straight line and a double generator.	Two sheaves of the first order.	"	5	6	XIII, say.
F_7	A double generator.	A sheaf of the first order.	"	—	—	—
F_2'	A triple line; no single line director.	A sheaf of the third order.	F_2	9	8	I.

Surfaces which Admit a Trinodal Quartic Section. $p = 0$.						
Species.	Character of the real nodal curve.	Forms constituted by real bitangent planes.	Reciprocal surface.	Corresponding species.		
				Cayley.	Cremona.	Salmon.
F_4	A triple line which is a generator.	A sheaf of the first order and one of the second order.	F_4	12	8	III.
—	A triple line, with a single line director.	A sheaf of the first order, each plane tritangent.	Of the same form.	3	9	II.
—	A triple line; not a generator.	A sheaf of the first order, each plane tritangent.	"	6	10	IV and V.
Surfaces which do not Admit a Trinodal Quartic Section. $p = 1$.						
—	Two distinct straight lines, but no double generator.	Two sheaves of the first order.	Of the same form.	1	11	XI, say.
—	One straight line, but no double generator.	One (doubly counting) sheaf of the first order.	"	4	12	XII, say.

From the above table it will be observed that the two self-reciprocal triple-line surfaces and the two surfaces which do not admit a trinodal quartic section (Cayley's third, sixth, first, and fourth species) have not been treated in these papers; and, further, that neither Cayley, Cremona, nor Salmon makes mention of the surface for which the two nodal lines are imaginary, although they all distinguish the surface for which these lines coincide from that for which the lines are real and distinct.

In a paper in volume XXI of this Journal, Dr. E. M. Blake has described simple mechanisms for generating many of the forms and subforms of the surfaces discussed in these papers, and I should add that it was through correspondence with him that my attention was called to the omissions in my former paper.

Northwestern University, Evanston, *August* 20, 1898.

Non-Euclidian Properties of Plane Cubics.

By Henry Freeman Stecker.

Take $\Omega_{xx}=0$ as the equation of the absolute; $\Omega_{xy}=0$ the equation of the polar of the point y with respect to the absolute; $C_{xx}=0$ the equation of the cubic. Then $\Omega_{ba}=0$ is the condition that the point b lie on the polar of the point a with respect to the absolute. Take (xy) as the non-Euclidian distance from the point x to the point y, or between the polars of those points with respect to the absolute, $\overline{xy}$, the corresponding distance from the point x to the polar of the point y with respect to the absolute.

Then we have:

$$\cos\frac{(xy)}{2ki}=\sin\frac{\overline{xy}}{2ki}=\frac{\Omega_{xy}}{\sqrt{\Omega_{xx}\,\Omega_{yy}}}.$$

It seems desirable to use, in any discussion of non-Euclidian properties of curves, certain terms used by W. K. Clifford and also by Professor Story in regard to certain properties of the conic, since they apply to any curve, viz. There are six intersections of the cubic with the absolute, the *absolute points.* Twelve common tangents to cubic and absolute, the *absolute tangents.* Six tangents to the cubic at the absolute points, the *asymptotes.* Fifteen lines joining the absolute points in pairs, the *focal lines.* Sixty-six intersections of pairs of absolute tangents, the *foci.* Twelve points of contact of the absolute tangents with the absolute, the *asymptotic points.* The lines joining the asymptotic points are the *directrices.* The intersections of the asymptotes are the *directors.*

Then there are certain other terms which apply to the cubic alone. Related to any pair of tangents there is a third tangent; viz., the tangent to the cubic at the third intersection with the cubic of the chord of contact of the two given tangents. Such third tangents will be spoken of as *third absolute tangents*, etc., depending upon the character of the original pair of tangents. Associated with three such tangents there is a satellite line, which will be designated as an *absolute satellite line*, etc. Then there are certain first and second polars, with respect to the cubic, of fixed points and lines. These are designated by prefixing the special name of the fixed point or line as *absolute tangent first polar*, etc. Lastly, *N. E. D.* is used to designate the non-Euclidian distance divided by the proper constant, i. e., $\frac{(xy)}{2ki}$.

Consider the six absolute points. Connect them by three straight lines, no point being on more than one of the lines. Then if a, b, c are the directors of these focal lines, their equations are:

$$\begin{aligned} \Omega_{xa} &= 0, \\ \Omega_{xb} &= 0, \\ \Omega_{xc} &= 0. \end{aligned}$$

Each of these lines cuts the cubic in a third point. These three points, say p_1, p_2, p_3, lie on a line, $C = 0$.

Then the equation of the cubic may be written in the three ways:

$$\begin{aligned} C_{xx} &\equiv \lambda_1 \Omega_{xa} \Omega_{xx} - C\Omega_{xb} \Omega_{xc} = 0, \\ C_{xx} &\equiv \lambda_2 \Omega_{xb} \Omega_{xx} - C\Omega_{xa} \Omega_{xc} = 0, \\ C_{xx} &\equiv \lambda_3 \Omega_{xc} \Omega_{xx} - C\Omega_{xa} \Omega_{xb} = 0, \end{aligned}$$

or what is the same thing,

$$\frac{\lambda_1}{C} = \frac{\Omega_{xb} \Omega_{xc}}{\Omega_{xa} \Omega_{xx}}, \tag{1}$$

$$\frac{\lambda_2}{C} = \frac{\Omega_{xa} \Omega_{xc}}{\Omega_{xb} \Omega_{xx}}, \tag{2}$$

$$\frac{\lambda_3}{C} = \frac{\Omega_{xa} \Omega_{xb}}{\Omega_{xc} \Omega_{xx}}. \tag{3}$$

But the right-hand member of (1) may be written:

$$\frac{\frac{\Omega_{xb}}{\sqrt{\Omega_{xx} \Omega_{bb}}} \cdot \frac{\Omega_{xc}}{\sqrt{\Omega_{xx} \Omega_{cc}}} \cdot \sqrt{\frac{\Omega_{bb} \Omega_{cc}}{\Omega_{aa} \Omega_{xx}}}}{\frac{\Omega_{xa}}{\sqrt{\Omega_{xx} \Omega_{aa}}}}.$$

Therefore, (1) may be written:

$$\frac{\lambda_1 \sqrt{\Omega_{aa}} \sqrt{\Omega_{xx}}}{C \cdot \sqrt{\Omega_{bb} \Omega_{cc}}} = \frac{\cos \frac{(xb)}{2ki} \cdot \cos \frac{(xc)}{2ki}}{\cos \frac{(xa)}{2ki}}, \tag{4}$$

similarly for (2) and (3).

$$\lambda_2 \frac{\sqrt{\Omega_{bb}}\sqrt{\Omega_{xx}}}{C\,.\sqrt{\Omega_{aa}\Omega_{cc}}} = \frac{\cos\frac{(xa)}{2ki}\cdot\cos\frac{(xc)}{2ki}}{\cos\frac{(xb)}{2ki}}, \tag{5}$$

$$\lambda_3 \frac{\sqrt{\Omega_{cc}}\sqrt{\Omega_{xx}}}{C\,.\sqrt{\Omega_{aa}\Omega_{bb}}} = \frac{\cos\frac{(xa)}{2ki},\ \cos\frac{(xb)}{2ki}}{\cos\frac{(xc)}{2ki}}. \tag{6}$$

Dividing (4) by (5) we have

$$\frac{\lambda_1}{\lambda_2}\cdot\frac{\Omega_{aa}}{\Omega_{bb}} = \frac{\cos^2\frac{(xb)}{2ki}}{\cos^2\frac{(xa)}{2ki}} = \frac{\sin^2\frac{\overline{xb}}{2ki}}{\sin^2\frac{\overline{xa}}{2ki}},$$

and similar relations from (4) and (6), (5) and (6). The left-hand member is constant, hence the theorem:

The ratio of the squares of the cosines (sines) of the N. E. D. from any point of a cubic to any pair of directors whose focal lines do not have a common absolute point (to any pair of focal lines not through the same absolute point) is constant.

If the lines are such that some of them pass through the same absolute point, then the above ratio multiplied by $\frac{C}{C'}$ is constant.

Take T and T' a pair of absolute tangents; H the corresponding third absolute tangent; Q the chord of contact and S the absolute satellite line. Then if the poles of these lines, with respect to the absolute, are represented by the corresponding small letters, we may write:

$$C_{xx} \equiv \Omega_{xt}\,\Omega_{xt'}\,\Omega_{xh} - \lambda\Omega^2_{xq}\,\Omega_{xs} = 0. \tag{7}$$

This gives

$$\frac{\frac{\Omega_{xt}}{\sqrt{\Omega_{xx}\Omega_{tt}}}\cdot\frac{\Omega_{xt'}}{\sqrt{\Omega_{xx}\Omega_{t't'}}}\cdot\frac{\Omega_{xh}}{\sqrt{\Omega_{xx}\Omega_{hh}}}}{\frac{\Omega^2_{xq}}{\Omega_{xx}\Omega_{qq}}\cdot\frac{\Omega_{xs}}{\sqrt{\Omega_{xx}\Omega_{ss}}}} = \lambda\sqrt{\frac{\Omega_{tt}\,\Omega_{t't'}\,\Omega_{hh}}{\Omega^2_{qq}\,\Omega_{ss}}},$$

or

$$\frac{\sin\frac{\overline{xt}}{2ki}\cdot\sin\frac{\overline{xt'}}{2ki}\cdot\sin\frac{\overline{xh}}{2ki}}{\sin^2\frac{\overline{xq}}{2ki}\cdot\sin\frac{\overline{xs}}{2ki}} = \text{const.} \tag{8}$$

Hence:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to any pair of absolute tangents and third absolute tangent, to that of the square of the sine of the N. E. D. to the chord of contact into the sine of the N. E. D. to the absolute satellite line, is constant.

For the absolute points we should have equation (7), and also the equation of the absolute, which may be written $TT' - \Omega_{xa}^2 = 0$, where a is the corresponding focus, simultaneously true. Hence, for such points, say a point m, we may write:

$$\Omega_{ma}^2 \, \Omega_{mb} = \lambda \Omega_{mq}^2 \, \Omega_{ms},$$

or

$$\left[\frac{\Omega_{ma}}{\Omega_{mq}}\right]^2 = \frac{\lambda \Omega_{ms}}{\Omega_{mh}},$$

which gives

$$\frac{\sin^2 \frac{\overline{ma}}{2ki}}{\sin^2 \frac{\overline{mq}}{2ki}} = \lambda \frac{\sin \frac{\overline{ms}}{2ki}}{\sin \frac{\overline{mh}}{2ki}}$$

as a relation connecting an absolute point m with the other quantities involved in the preceding theorem.

Take a_1, a_2, a_3 the poles, with respect to the absolute, of the tangents at three collinear points of inflexion; c that of their chord of contact.

Then we may write:

$$C_{xx} \equiv \Omega_{xa_1} \cdot \Omega_{xa_2} : \Omega_{xa_3} - \lambda \Omega_{xc}^3 = 0,$$

or

$$\frac{\frac{\Omega_{xa_1}}{\sqrt{\Omega_{xx}\,\Omega_{a_1a_1}}} \cdot \frac{\Omega_{xa_2}}{\sqrt{\Omega_{xx}\,\Omega_{a_2a_2}}} \cdot \frac{\Omega_{xa_3}}{\sqrt{\Omega_{xx}\,\Omega_{a_3a_3}}}}{\frac{\Omega_{xc}^3}{(\sqrt{\Omega_{xx}\,\Omega_{cc}})^3}} = \lambda \frac{\sqrt{\Omega_{a_1a_1}\,\Omega_{a_2a_2}\,\Omega_{a_3a_3}}}{\Omega_{cc}^{\frac{3}{2}}} = \text{const.}$$

This gives

$$\frac{\sin \frac{\overline{xa_1}}{2ki} \cdot \sin \frac{\overline{xa_2}}{2ki} \cdot \sin \frac{\overline{xa_3}}{2ki}}{\sin^3 \frac{\overline{xc}}{2ki}} = \text{const.}$$

Hence the theorem:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to the tangents at three collinear points of inflexion, to the cube of the sine of the N. E. D. to the chord of contact is constant.

If $\Omega_{xa}=0$ and $\Omega_{xb}=0$ are two lines cutting the cubic in three sets of points; $\Omega_{xa_1}=0$, $\Omega_{xa_2}=0$ and $\Omega_{xa_3}=0$ be lines joining corresponding pairs of these points; and $\Omega_{xc}=0$ be the line on which the remaining intersections must lie, then we may write:

$$C_{xx}\equiv\Omega_{xa_1}\Omega_{xa_2}\Omega_{xa_3}-k\Omega_{xa}\Omega_{xb}\Omega_{xc}=0.$$

Treating this as before, we have

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xa_3}}{2ki}}{\sin\frac{\overline{xa}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}}=k\sqrt{\frac{\Omega_{a_1a_1}\Omega_{a_2a_2}\Omega_{a_3a_3}}{\Omega_{aa}\Omega_{bb}\Omega_{cc}}}. \tag{A}$$

Since all the lines are fixed if $\Omega_{xa}=0$ and $\Omega_{xb}=0$, therefore, the left-hand ratio is constant for any two fixed lines $\Omega_{xa}=0$ and $\Omega_{xb}=0$. This relation applies best to focal lines—since, then, two of the chords of contact are also focal lines—as follows:

Consider four focal lines through four absolute points—two through each point. Say Ω_{xa_1}, Ω_{xa_2}, Ω_{xa_3}, Ω_{xa_4}, where a_1, a_2, a_3, a_4 are the corresponding directors; take Ω_{xb} and Ω_{xc} the chords of contact of the third intersections of pairs of focal lines not through a common point. Then from relation (A) we write:

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xa_3}}{2ki}\cdot\sin\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}}=\text{const.} \tag{B}$$

Hence:

The ratio of the sines (cosines) of the N. E. D. between any point of a cubic and any pair of focal lines, not through the same absolute point, and between the chord of the third intersections of such a pair (to the directors of such a pair of focal lines and to the pole, with respect to the absolute, of the chord of third intersections) to the corresponding product for any other like pair of focal lines, is constant.

If we consider the six possible focal lines through four absolute points, there would be three concurrent lines, $\Omega_{xb}=0$, $\Omega_{xc}=0$, $\Omega_{xd}=0$, where the first two have the same meaning as above and $\Omega_{xd}=0$ is the corresponding sine for the other pair of focal lines. We should find six relations like (B), but only two of them independent, viz. (B) and (C) where (C) equals

$$\frac{\sin\frac{\overline{xa_5}}{2ki}\cdot\sin\frac{\overline{xa_6}}{2ki}\cdot\sin\frac{\overline{xb}}{2ki}}{\sin\frac{\overline{xa_3}}{2ki}\cdot\sin\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xd}}{2ki}}=\text{const.} \tag{C}$$

where a_i $(i=1,\ldots.6)$ are the directors of the six focal lines.

The product of these two independent relations gives a relation *between the six focal lines through any four absolute points and the three concurrent lines on which their third intersections lie, viz.*

$$\frac{\sin\frac{\overline{xa_1}}{2ki}\cdot\sin\frac{\overline{xa_2}}{2ki}\cdot\sin\frac{\overline{xa_5}}{2ki}\cdot\sin\frac{\overline{xa_6}}{2ki}\cdot\sin^2\frac{\overline{xb}}{2ki}}{\sin^2\frac{\overline{xa_3}}{2ki}\cdot\sin^2\frac{\overline{xa_4}}{2ki}\cdot\sin\frac{\overline{xc}}{2ki}\cdot\sin\frac{\overline{xd}}{2ki}}=\text{const.}\qquad\text{(D)}$$

Since the chords joining asymptotic points are directrices, it follows that relations corresponding to (B), (C) and (D) hold for directrices, if we replace focal lines of original pair by directrices, directors by foci and absolute points by the intersections of the directrix with the cubic; here, evidently, the chords will not be directrices.

If we consider a pair of asymptotes, we may use equation (8). We have the two tangents as asymptotes, and the chord of contact is a focal line. Then we have the third asymptotic tangent and asymptotic satellite line. We have then at once the relation:

The ratio of the product of the sines of the N. E. D. from any point of a cubic to any pair of asymptotic tangents and third asymptotic tangent to the product of the square of the sine of the N. E. D. to the corresponding focal line into the sine of the N. E. D. to the asymptotic satellite line, is constant.

Consider the triangle with vertices at any two directors and any point of the cubic. Take a and b the lines joining the point of the cubic to the directors; c the line joining the directors; α, β, γ their opposite angles. Then, from the first theorem of this paper we have:

$$\frac{\cos^2\frac{\overline{a}}{2ki}}{\cos^2\frac{\overline{b}}{2ki}}=\lambda;\ \text{the relation}\ \frac{\sin^2\frac{\overline{a}}{2ki}}{\sin^2\frac{\overline{b}}{2ki}}=\frac{\sin^2\frac{\overline{\alpha}}{2k'}}{\sin^2\frac{\overline{\beta}}{2k'}}\ \text{is also true, which gives}$$

$$\frac{1-\cos^2\frac{\overline{a}}{2ki}}{1-\cos^2\frac{\overline{b}}{2ki}}=\frac{\dfrac{1}{\cos^2\frac{\overline{b}}{2ki}}-\dfrac{\cos^2\frac{\overline{a}}{2ki}}{\cos^2\frac{\overline{b}}{2ki}}}{\dfrac{1}{\cos^2\frac{\overline{b}}{2ki}}-1}=\frac{\sec^2\frac{\overline{b}}{2ki}-\lambda}{\sec^2\frac{\overline{b}}{2ki}-1}=\frac{\sin^2\frac{\overline{\alpha}}{2k'}}{\sin^2\frac{\overline{\beta}}{2k'}}$$

This gives
$$\frac{\sin^2 \frac{\beta}{2k'} - \sin^2 \frac{\alpha}{2k'} \cdot \sin^2 \frac{b}{2ki}}{\cos^2 \frac{b}{2ki} \cdot \sin^2 \frac{\beta}{2k'}} = \text{const.}$$

Similarly, we obtain the corresponding relation involving a, viz.
$$\frac{\sin^2 \frac{\alpha}{2k'} - \sin^2 \frac{\beta}{2k'} \sin^2 \frac{a}{2ki}}{\cos^2 \frac{a}{2ki} \sin^2 \frac{\alpha}{2k'}} = \text{const.}$$

These constants are reciprocals, hence we have :

If a and b are the non-Euclidian distances from any point of a cubic to any pair of directors and α, β the angles which they make with the line joining the directors, then we have the relation:
$$\begin{vmatrix} \sin^2 \frac{\alpha}{2k'} & \sin^2 \frac{\beta}{2k'} \\ \sin^2 \frac{a}{2ki} & 1 \end{vmatrix} \cdot \begin{vmatrix} \sin^2 \frac{\beta}{2k'} & \sin^2 \frac{\alpha}{2k'} \\ \sin^2 \frac{b}{2ki} & 1 \end{vmatrix} = \cos^2 \frac{a}{2ki} \cdot \cos^2 \frac{b}{2ki} \cdot \sin^2 \frac{\alpha}{2k'} \cdot \sin^2 \frac{\beta}{2k'} \,.$$

We also have the relation:
$$\cos \frac{a}{2ki} = \cos \frac{b}{2ki} \cos \frac{c}{2ki} + \cos \frac{\alpha}{2k'} \sin \frac{a}{2ki} \sin \frac{b}{2ki} \,. \quad \frac{\cos \frac{a}{2ki}}{\cos \frac{b}{2ki}} \text{ and } \cos \frac{c}{2ki}$$
are constant, hence we may write: $\cos \frac{\alpha}{2k'} \sin \frac{a}{2ki} \tan \frac{b}{2ki} = \text{const.}$, and similarly, $\cos \frac{\beta}{2k'} \sin \frac{b}{2ki} \tan \frac{a}{2ki} = \text{const.}$, which may be written,
$$\left\{ \begin{aligned} &\sin \frac{a}{2ki} \sin \frac{b}{2ki} \frac{\cos \frac{\alpha}{2k'}}{\cos \frac{b}{2ki}} = \text{const.}, \\ &\sin \frac{a}{2ki} \sin \frac{b}{2ki} \frac{\cos \frac{\beta}{2k'}}{\cos \frac{a}{2ki}} = \text{const.} \end{aligned} \right.$$

Hence, we may say :

The product of the sines of the N. E. D. from any point of a cubic to any pair *of directors multiplied by the ratio of the cosine of the angle at either director to the cosine of the N. E. D. of the adjacent side, is constant.*

We may write:

$$\sin\frac{c}{2ki} - \cos\frac{c}{2ki}\tan\frac{b}{2ki}\cos\frac{\alpha}{2k'} = \lambda\sin\frac{\alpha}{2k'}\cos\frac{\beta}{2k'},$$

$$\sin\frac{c}{2ki} - \cos\frac{c}{2ki}\tan\frac{a}{2ki}\cos\frac{\beta}{2k'} = \frac{1}{\lambda}\sin\frac{\beta}{2k'}\cos\frac{\alpha}{2k'},$$

where λ is constant and equal to $\dfrac{\sin\frac{a}{2ki}}{\cos\frac{a}{2ki}}$; calling $\sin\frac{c}{2ki} = k$ and $\cos\frac{c}{2ki} = h$, we have by subtraction:

$$\tan\frac{a}{2ki}\cdot\cos\frac{\beta}{2k'} - \tan\frac{b}{2ki}\cos\frac{\alpha}{2k'}$$
$$= \frac{1}{h}\left[\sin\frac{\alpha}{2k'}\cos\frac{\beta}{2k'} - \frac{1}{\lambda}\sin\frac{\beta}{2k'}\cos\frac{\alpha}{2k'}\right].$$

Hence, we have: *The ratio* $\dfrac{\begin{vmatrix} \tan\frac{a}{2ki} & \cos\frac{\alpha}{2k'} \\ \tan\frac{b}{2ki} & \cos\frac{\beta}{2k'} \end{vmatrix}}{\begin{vmatrix} \lambda\sin\frac{\alpha}{2k'} & \cos\frac{\alpha}{2k'} \\ \frac{1}{\lambda}\sin\frac{\beta}{2k'} & \cos\frac{\beta}{2k'} \end{vmatrix}}$ *is constant for any point of a cubic and any pair of directors.*

Starting with the first theorem of this paper, and taking a and b to be the distances from any point of a cubic to any pair of focal lines not concurrent at the same absolute point; c the distance between their intersections with the pair of focal lines, and α, β, γ the opposite angles. We may write at once:

$$\frac{\sin^2\frac{\alpha}{2k'}}{\sin^2\frac{\beta}{2k'}} = \frac{\sin^2\frac{a}{2ki}}{\sin^2\frac{b}{2ki}} = \text{const.}$$

Hence:

The ratio of the squares of the sines of the angles which the lines from any point *of a cubic to any pair of focal lines, not concurrent at the same absolute point, make with the line joining the intersections, is constant.*

In relation (8), join the intersection of $\overline{xt}$ with its absolute tangent to the intersection of $\overline{xy}$ with the chord of contact and call the angles thus formed at the intersections δ and α; connect the last intersection with the intersection or $\overline{xt'}$ with its tangent and call the angles β, δ_1; lastly, join the intersection of $\overline{xh}$ with its tangent to the intersection of $\overline{xs}$ with the satellite and call the angles γ, π. Then we have the relation:

$$\frac{\sin\frac{\alpha}{2k'} \cdot \sin\frac{\beta}{2k'} \cdot \sin\frac{\gamma}{2k'}}{\sin\frac{\delta}{2k'} \cdot \sin\frac{\delta_1}{2k'} \cdot \sin\frac{\pi}{2k'}} = \text{const.}$$

for all points of the cubic.

Evidently other relations involving angles—and for (B) of page 35 containing other constants—could be derived in a similar manner, but they would not be simple, and will not be deduced here.

I wish next to derive two or three properties of a cubic from certain non-Euclidian properties of a conic. Such properties of a conic are taken from Professor Story's paper before mentioned. Other properties might be derived in a similar manner.

The relation holds for the conic that the product of the sines of the N. E. D. from any point of the conic to the focal lines of either pair is constant.

That constant is really the parameter of a pencil of conics through four of the absolute points.

If we take as the base-points of the pencil of conics four of the absolute points and as the fixed conics a pair of focal lines and the absolute, the cubic will be generated by this pencil of conics and a pencil of rays. This pencil of conics and the cubic will have the same set of focal lines. The cubic will have a point on each conic of the pencil, hence the product of the sines of the N. E. D., from any point of the cubic to its focal lines of either pair, varies as the parameter of the pencil of conics of which the pair of focal lines is one fixed conic and the absolute the other.

If k_1 and k_2 are the parameters for two different pairs of focal lines, we may write:

The product of the sines of N. E. D. from any point of a cubic to the focal lines of any pair equals $\frac{k_1}{k_2}$ *times the corresponding product for any other pair.*

Next, consider the tangent to a cubic. It will be tangent to one of the conics of the pencil through four of the absolute points; for the pencil contains all the conics through the four points, and a conic through four points and tangent to a given line is possible.

As the tangent to the cubic varies, the conic will vary, but its focal lines and directors, which are also focal lines and directors for the cubic, will not vary. Considering, then, two different focal lines and the directors of those lines, we may, from a property of the conic, write:

The ratio of the sines of the N. E. D. from any tangent to a cubic to any pair of focal lines equals $\frac{k_1}{k_2}$ *times the ratio of the sines of the N. E. D. to the directors of those focal lines.*

Consider any absolute tangent to a cubic, and the polar conic, with respect to the cubic, of the point of contact. This conic will pass through the point of contact and have the same absolute tangent; hence, from a property of a conic, we have:

Any point of contact of the cubic with an absolute tangent is equidistant from the intersections with those tangents of any pair of focal lines of the polar conic of the point of contact. Also, the absolute tangent makes equal angles with the lines joining the point of contact to either pair of foci of the polar conic of the point of contact.

This brings us to the consideration of first and second polars which I reserve for a future paper.

Note on the Differential Invariants of Goursat and Painlevé.

By E. O. Lovett.

In a memoir on linear homogeneous differential equations published in the last number of the Journal de l'École Polytechnique, second series, fourth cahier, M. A. Boulanger constructs a class of differential invariants first studied by MM. Goursat and Painlevé for ternary and quaternary finite linear groups. In deriving the forms of these invariants for quaternary groups, the author makes use of the process of elimination employed by Sophus Lie* for the construction of differential invariants when the finite equations of a group are given. The method of procedure followed by M. Boulanger leads readily to a class of differential invariants under the n-ary finite linear group, as in fact he himself predicted.

Consider a finite group of linear substitutions in n non-homogeneous variables u_i and the n corresponding fundamental invariant functions

$$x_i = f_i(u_1, u_2, \dots, u_n), \qquad i = 1, 2, \dots, n. \tag{1}$$

Let the defining equations of the group be

$$u_i = \frac{a_{i,1}u_1' + a_{i,2}u_2' + \dots + a_{i,n}u_n' + a_{i,n+1}}{a_{0,1}u_1' + a_{0,2}u_2' + \dots + a_{0,n}u_n' + a_{0,n+1}}, \qquad i = 1, \dots, n, \tag{2}$$

which, for convenience, are written,

$$\left.\begin{aligned} a_{0,1}u_1u_1' + a_{0,2}u_1u_2' + \dots + a_{0,n}u_1u_n' + a_{0,n+1}u_1 \qquad & \\ = a_{1,1}u_1' + a_{1,2}u_2' + \dots + a_{1,n}u_n' + a_{1,n+1}&, \\ a_{0,1}u_2u_1' + a_{0,2}u_2u_2' + \dots + a_{0,n}u_2u_n' + a_{0,n+1}u_2 \qquad & \\ = a_{2,1}u_1' + a_{2,2}u_2' + \dots + a_{2,n}u_n' + a_{2,n+1}&, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{0,1}u_nu_1' + a_{0,2}u_nu_2' + \dots + a_{0,n}u_nu_n' + a_{0,n+1}u_n \qquad & \\ = a_{n,1}u_1' + a_{n,2}u_2' + \dots + a_{n,n}u_n' + a_{n,n+1}&. \end{aligned}\right\} \tag{3}$$

---

* See Lie, "Theorie der Transformationsgruppen," Unter Mitwirkung von F. Engel, volume I, chapters XIII and XXV.

By differentiating the first equation of this system (3) successively with regard to $x_1, x_2, \ldots\ldots, x_n$, we have

$$\left.\begin{aligned}
&(a_{0,1}u_1' + u_{0,2}u_2' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1})\frac{\partial u_1}{\partial x_1} \\
&\qquad\qquad = (a_{1,1} - a_{0,1}u_1)\frac{\partial u_1'}{\partial x_1} + \ldots\ldots + (a_{1,n} - a_{0,n})\frac{\partial u_n'}{\partial x_1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&(a_{0,1}u_1' + a_{0,2}u_2' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1})\frac{\partial u_1}{\partial x_n} \\
&\qquad\qquad = (a_{1,1} - a_{0,1}u_1)\frac{\partial u_1'}{\partial x_n} + \ldots\ldots + (a_{1,n} - a_{0,n})\frac{\partial u_n'}{\partial x_n}.
\end{aligned}\right\} \quad (4)$$

From this system (4) we deduce by solution that

$$\left.\begin{aligned}
a_{1,1} &= a_{0,1}u_1 + \frac{D(u_1, u_2', \ldots\ldots, u_n')}{D(u_1, u_2, \ldots\ldots, u_n)}(a_{0,1}u_1' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1}), \\
a_{1,2} &= a_{0,2}u_1 + \frac{D(u_1', u_1, u_3', \ldots\ldots, u_n')}{D(u_1, u_2, \ldots\ldots, u_n)}(a_{0,1}u_1' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1}), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{1,n} &= a_{0,n}u_1 + \frac{D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1)}{D(u_1, u_2, \ldots\ldots, u_n)}(a_{0,1}u_1' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1}),
\end{aligned}\right\} \quad (5)$$

where $D(u_1', u_2', \ldots\ldots, u_n')$ represents the functional determinant of $u_1', \ldots\ldots, u_n'$ with regard to $x_1, \ldots\ldots, x_n$.

Differentiating this same first equation of the system (3), first twice with regard to x_1, and then with regard to x_1 and x_2, and substituting the above values of $a_{1,1}, a_{1,2}, \ldots\ldots, a_{1,n}$ in the results, we obtain

$$\left.\begin{aligned}
&2\left(a_{0,1}\frac{\partial u_1'}{\partial x_1} + \ldots\ldots, + a_{0,n}\frac{\partial u_n'}{\partial x_1}\right)\frac{\partial u_1}{\partial x_1} \\
&\quad = \frac{a_{0,1}u_1' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1}}{D(u_1, u_2, \ldots\ldots, u_n)}\left\{-\frac{\partial^2 u_1}{\partial x_1^2}D(u_1', \ldots\ldots, u_n')\right. \\
&\quad\left. + \frac{\partial^2 u_1'}{\partial x_1^2}D(u_1, u_2', \ldots\ldots, u_n') + \ldots\ldots + \frac{\partial^2 u_n'}{\partial x_1^2}D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1)\right\}, \\
&\left(a_{0,1}\frac{\partial u_1'}{\partial x_1} + \ldots\ldots + a_{0,n}\frac{\partial u_n'}{\partial x_1}\right)\frac{\partial u_1}{\partial x_2} + \left(a_{0,1}\frac{\partial u_1'}{\partial x_2} + \ldots\ldots + a_{0,n}\frac{\partial u_n'}{\partial x_2}\right)\frac{\partial u_1}{\partial x_1} \\
&\quad = \frac{a_{0,1}u_1' + \ldots\ldots + a_{0,n}u_n' + a_{0,n+1}}{D(u_1, u_2, \ldots\ldots, u_n)}\left\{-\frac{\partial^2 u_1'}{\partial x_1\,\partial x_2}D(u_1', \ldots\ldots, u_n')\right. \\
&\quad\left. + \frac{\partial^2 u_1}{\partial x_1\,\partial x_2}D(u_1, u_2', \ldots\ldots, u_n') + \ldots\ldots + \frac{\partial^2 u_n}{\partial x_1\partial x_2}D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1)\right\}.
\end{aligned}\right\} \quad (6)$$

For convenience put

$$(a_{0,1}u_1' + \dots\dots + a_{0,n}u_n' + a_{0,n+1})\,\xi_i = \left(a_{0,1}\frac{\partial u_1'}{\partial x_i} + \dots\dots + a_{0,n}\frac{\partial u_n'}{\partial x_i}\right) D(u_1', u_2', \dots\dots, u_n').$$

Then the above equations (6) become

$$\left.\begin{array}{l}
2\xi_1\dfrac{\partial u_1}{\partial x_1} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_1^2} & \dfrac{\partial^2 u_1'}{\partial x_1^2} & \dfrac{\partial^2 u_2'}{\partial x_1^2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_1^2} \\ \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_n'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0, \\[4em]
\xi_1\dfrac{\partial u_1}{\partial x_2} + \xi_2\dfrac{\partial u_1}{\partial x_1} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_1\partial x_2} & \dfrac{\partial^2 u_1'}{\partial x_1\partial x_2} & \dfrac{\partial^2 u_2'}{\partial x_1\partial x_2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_1\partial x_2} \\ \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_2'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0;
\end{array}\right\} \qquad (7)$$

in the same way, by differentiating the first of equations (3) twice with regard to the other variables, we obtain $\frac{1}{2}\{n(n+1) - 4\}$ other equations,

$$2\xi_2\frac{\partial u_1}{\partial x_2} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_2^2} & \dfrac{\partial^2 u_1'}{\partial x_2^2} & \dfrac{\partial^2 u_2'}{\partial x_2^2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_2^2} \\ \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_2'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0, \dots\dots; \qquad (8)$$

and by treating the remaining equations of (3) one by one in like manner, we have, all told, $\frac{1}{2}n^2(n+1)$ equations from which the n functions $\xi_1, \xi_2, \dots\dots, \xi_n$ may be eliminated. The resulting $\frac{1}{2}n(n-1)(n+2)$ equations define as many differential invariants which come to light as follows.

The functions ξ_1 and ξ_2 can be eliminated from the three equations written above by multiplying the first by $\left(\frac{\partial u_1}{\partial x_2}\right)^2$, the second by $-\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}$, the third by $\left(\frac{\partial u_1}{\partial x_1}\right)^2$, and adding member to member. We thus have

$$\left\{\frac{\partial^2 u_1}{\partial x_1^2}\left(\frac{\partial u_1}{\partial x_2}\right)^2 - 2\,\frac{\partial u_1}{\partial x_1}\,\frac{\partial u_1}{\partial x_2}\,\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\} D(u_1', u_2', \ldots, u_n')$$
$$-\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2^2}\right)\left\{\left|\frac{\partial^2 u_1'}{\partial x_1^2}, \frac{\partial u_2'}{\partial x_2}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right| - 2\left|\frac{\partial^2 u_1'}{\partial x_1 \partial x_2}, \frac{\partial u_2'}{\partial x_1}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right|\right\}$$
$$+\left(\frac{\partial u_1}{\partial x_2}\right)^2\left|\frac{\partial^2 u_1'}{\partial x_1^2}, \frac{\partial u_2'}{\partial x_1}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right|$$
$$-\left(\frac{\partial u_1}{\partial x_1}\right)^2\left|\frac{\partial^2 u_1'}{\partial x_2^2}, \frac{\partial u_2'}{\partial x_2}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right| + \ldots = 0, \tag{9}$$

The ratios of any two of the coefficients of this equation (in which coefficients only the derivatives of $u_1', u_2', \ldots, u_n'$ enter) are not changed by a linear substitution.

By forming the equations analogous to (9), we obtain the following system of differential invariants, omitting the factor $1/D(u_1', \ldots, u_n')$ and writing

$$\left|\frac{\partial^2 u_1'}{\partial x_h \partial x_k}, \frac{\partial u_2'}{\partial x_l}, \frac{\partial u_3'}{\partial x_m}, \ldots, \frac{\partial u_n'}{\partial x_\mu}\right| \equiv \{x_h x_k, x_l, x_m, \ldots, x_\mu\},$$

namely, $n(n-1)$ of the form

$$\{x_i x_i,\ x_i,\ x_j, \ldots, x_\mu\}, \tag{10}$$

where

$$j \neq \mu \neq i; \qquad i = 1, 2, \ldots, n,$$

and $x_j, \ldots, x_\mu$ is a cyclical combination of the remaining $n-1$ taken $n-2$ at a time; $\frac{1}{2}n(n-1)(n-2)$ of the form

$$\{x_i x_j,\ x_i,\ x_j,\ x_k, \ldots, x_\mu\}, \tag{11}$$

where

$$j \neq k \neq \mu \neq i; \quad i = 1, \ldots, n; \ j = 1, \ldots, n;$$

and $x_k, \ldots, x_\mu$ is a cyclical combination of the remaining; and $n(n-1)$ of the form

$$\{x_{ii},\ x_j,\ x_{j+1}, \ldots, x_{i-1}\} - 2\{x_i x_j,\ x_i,\ x_{j+1}, \ldots, x_{i-1}\}, \tag{12}$$
$$i = 1, \ldots, n; \ j = 1, \ldots, n.$$

This system of invariants (10), (11), (12) may be regarded as an extension of Schwarz's invariant

$$\frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

to the case of n variables.

For $n = 2$, the system (10), (11), (12) gives the system set up by M. Goursat (Comptes Rendus, vol. 104, p. 1362),

$$\frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial y^2} = A\Delta, \qquad \frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial x^2} = D\Delta,$$

$$\frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial x\,\partial y}\right) = B\Delta,$$

$$\frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial x\,\partial y}\right) = C\Delta,$$

where

$$\Delta = \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\,\frac{\partial u}{\partial y}.$$

For $n = 3$, we have the system constructed by Painlevé (Comptes Rendus, vol. 104, p. 1500),

$$\begin{array}{lll} |T_{xy},\ U_x,\ V_y|, & |T_{yz},\ U_y,\ V_z|; & |T_{zx},\ U_z,\ V_x|, \\ |T_{xx},\ U_x,\ V_y|, & |T_{xx},\ U_x,\ V_z|, & |T_{yy},\ U_y,\ V_z|, \\ |T_{yy},\ U_y,\ V_x|, & |T_{zz},\ U_z,\ V_x|, & |T_{zz},\ U_z,\ V_y|. \end{array}$$

$$\begin{array}{ll} |T_{xx},\ U_y,\ V_z| - 2|T_{xy},\ U_x,\ V_z|; & |T_{xx},\ U_y,\ V_z| - 2|T_{xz},\ U_y,\ V_x|, \\ |T_{yy},\ U_z,\ V_x| - 2|T_{xy},\ U_z,\ V_y|, & |T_{yy},\ U_z,\ V_y| - 2|T_{xz},\ U_y,\ V_x|, \\ |T_{xz},\ U_x,\ V_y| - 2|T_{zx},\ U_z,\ V_y|, & |T_{zz},\ U_x,\ V_y| - 2|T_{xy},\ U_x,\ V_z|, \end{array}$$

where the factor $1/D(T,\ U,\ V)$ has been omitted.

PRINCETON, NEW JERSEY, *May* 16, 1899.

Supplementary Note on Projective Invariants.

By E. O. Lovett.

While a note on projective invariants, inserted in the April number of volume XXI of this Journal was printing, Lie's method for determining differential invariants was applied to the construction* of the following more general second order differential projective invariants of a system of $m+1$ points in a space of $n+1$ dimensions, namely,

$$\sum_{0}^{m}{}_{i}\frac{\left|p^{(k)}_{1,1},\, p^{(k)}_{2,2},\, \ldots\ldots,\, p^{(k)}_{n,n}\right|}{\left|p^{(i)}_{1,1},\, p^{(i)}_{2,2},\, \ldots\ldots,\, p^{(i)}_{n,n}\right|}\left\{\frac{x_0^{(i)}-x_0^{(k)}-\sum_{1}^{n}{}_{j}\,p_j^{(i)}\left(x_j^{(i)}-x_j^{(k)}\right)}{x_0^{(i)}-x_0^{(k)}-\sum_{1}^{n}{}_{j}\,p_j^{(k)}\left(x_j^{(i)}-x_j^{(k)}\right)}\right\}^{n+2}, \tag{1}$$

$$k=0,\,1,\,2,\,\ldots\ldots,\,m,$$

where $x_0^{(i)},\, x_1^{(i)},\, x_2^{(i)},\, \ldots\ldots,\, x_n^{(i)}, \qquad i=0,\,1,\,2,\,\ldots\ldots,\,m$ (2)

are the coordinates of the system of points, and

$$p_j^{(i)}=\frac{\partial x_0^{(i)}}{\partial x_j},\quad p_{i,j}=\frac{\partial^2 x_0^{(i)}}{\partial x_i\,\partial x_j}=p_{j,i}. \tag{3}$$

For the case $m=1$, the invariants reduce to the single one

$$\frac{\left|p^{(k)}_{1,1},\, \ldots\ldots,\, p^{(k)}_{n,n}\right|}{\left|p^{(i)}_{1,1},\, \ldots\ldots,\, p^{(i)}_{n,n}\right|}\left\{\frac{x_0^{(i)}-x_0^{(k)}-\sum_{1}^{n}{}_{j}\,p_j^{(i)}\left(x_j^{(i)}-x_j^{(k)}\right)}{x_0^{(i)}-x_0^{(k)}-\sum_{1}^{n}{}_{j}\,p_j^{(k)}\left(x_j^{(i)}-x_j^{(k)}\right)}\right\}^{n+2}; \tag{4}$$

---

* See Comptes Rendus, August 16, 1898; Darboux's Bulletin des sciences mathématiques, January, 1899.

moreover, the invariance of the forms (1) follows *a fortiori* by summation from the invariance of (4).

In the note referred to above, the cases of (1) corresponding to $m = r$, $n = 1, 2$ were applied to the construction of certain geometrical theorems.

The particular cases of (4) corresponding to $m = 1$, $n = 1, 2$ were given by Mr. Bouton;[*] as remarked for the general case, the invariance of (1) for $m = r$, $n = 1, 2$ follows *a fortiori* from the invariance of the form (4) corresponding to $m = 1$, $n = 1, 2$.

By noting the relation[†]

$$(-1)^n |p^{(i)}_{1,1}, \ldots\ldots, p^{(i)}_{n,n}| = G_i \left\{ 1 + \sum_1^n (p^{(i)}_j)^2 \right\}^{\frac{n}{2}+1}, \tag{5}$$

where G_i is the Gaussian curvature as generalized by Kronecker,[‡] the above invariants may be given the following geometrical interpretation:

Take $m + 1$ hypersurfaces arbitrarily chosen, except that through every one of the $m + 1$ points of the system, there should pass at least one hypersurface; let P_i, P_k, H_i, H_k be respectively the points $(x^{(i)}_0, \ldots\ldots, x^{(i)}_n)$, $(x^{(k)}_0, \ldots\ldots, x^{(k)}_n)$, and the hypersurfaces through these points; join P_k by straight lines to all the other points of the system; let θ_i be the angle between the normal to H_k at P_k and the straight line $P_i P_k$; ϕ_i the angle between the line $P_i P_k$ and the normal to H_i at P_i; finally, let $\rho_{j,1}, \rho_{j,2}, \ldots\ldots, \rho_{j,n}$ be the principal radii of curvature of H_j at P_j; then the invariance of the expressions (1) shows that the forms

$$\sum_0^m{}_i \left\{ \prod_{j=1}^{j=n} \rho_{k,j} \cos^{n+2}\theta_i \middle/ \prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2}\phi_i \right\}, \tag{6}$$
$$k = 0, 1, 2, \ldots\ldots, m$$

suffer no change by projective transformation; and in particular it follows from the invariance of (4) that every element of the last summation is an absolute constant.

By employing the investigations[§] of Casorati and von Lilienthal in the theory of curvature, the geometric interpretation of the above invariants can be varied; these latter forms are simple enough for space of three dimensions, but become very complicated for spaces of higher dimensions; this fact is another confirmation of Darboux's dictum,* that the total curvature is the most important notion of curvature for geometry.

---

[*] Bulletin of the American Mathematical Society, April, 1898.

[†] Beez, Mathematische Annalen, Bd. VII.

[‡] Berichte der Berliner Akademie der Wissenschaften, 1869.

[§] Acta Mathematica, tomes XIV, XVI.

The invariants (1) and (4) are productive of various geometrical theorems for which the reader is referred to the notes already cited.

---

* "Théorie des Surfaces," t. II, p. 365.

PRINCETON, NEW JERSEY, *May* 16, 1899.

Certain Subgroups of the Betti-Mathieu Group.

By L. E. Dickson.

1. It was shown in the writer's Dissertation[*] that the transformation in one variable,

$$X' = \sum_{i=1}^{m} A_i X^{p^{n(m-i)}} \tag{A}$$

represents a substitution upon the marks of the Galois Field of order p^{nm} if, and only if, the determinant

$$|A| \equiv \begin{vmatrix} A_1 & A_2 & \dots\dots & A_m \\ A_2^{p^n} & A_3^{p^n} & \dots\dots & A_1^{p^n} \\ A_3^{p^{2n}} & A_4^{p^{2n}} & \dots\dots & A_2^{p^{2n}} \\ \dots & \dots & \dots & \dots \\ A_m^{p^{n(m-1)}} & A_1^{p^{n(m-1)}} & \dots\dots & A_{m-1}^{p^{n(m-1)}} \end{vmatrix}$$

does not vanish in the Field. The totality of substitutions (A) form a group studied by Betti (for $n = 1$) and by Mathieu. This Betti-Mathieu Group was proven in the dissertation cited to be identical with Jordan's group of all linear homogeneous substitutions on m indices,

$$\xi_i' = \sum_{j=1}^{m} \alpha_{ij} \xi_j \qquad (i = 1, \dots\dots, m)$$

belonging the $GF[p^n]$. In setting up certain subgroups of the Betti-Mathieu Group, we make use of the following formula, holding for any integer k and quantity X such that $X^{p^{nm}} \equiv X$:

$$\left\{\sum_{i=1}^{m} A_i X^{p^{n(m-i)}}\right\}^{p^{nk}} \equiv \sum_{i=1}^{m} A_{i+k}^{p^{nk}} X^{p^{n(m-i)}}, \qquad (\text{mod } p), \tag{1}$$

where the subscripts to A_{i+k} are taken modulo m.

---

[*] Annals of Mathematics, pp. 65–120, and pp. 161–183, 1897.

2. Consider the subgroup of the Betti-Mathieu Group defined by the relative invariant, in which B belongs to the $GF\,[p^{nm}]$,

$$Z \equiv \sum_{j=0}^{m-1} (BX)^{p^{nj}}.$$

Applying to Z the substitution (A), we have, by (1),

$$Z' \equiv \sum_{j=0}^{m-1} (BX')^{p^{nj}} = \sum_{j=0}^{m-1} \left\{ B^{p^{nj}} \sum_{i=1}^{m} A_{i+j}^{p^{nj}} X^{p^{n(m-i)}} \right\}.$$

The conditions for the identity $Z' = \rho Z$ are, therefore,

$$\sum_{j=0}^{m-1} B^{p^{nj}} A_{i+j}^{p^{nj}} = \rho B^{p^{n(m-i)}}, \qquad (i = 1, 2, \ldots, m). \qquad (2)$$

Raising (2) to the power p^n and setting $l = j + 1$, we find

$$\sum_{l=1}^{m} B^{p^{nl}} A_{i+l-1}^{p^{nl}} \equiv \sum_{l=1}^{m-1} B^{p^{nl}} A_{i+l-1}^{p^{nl}} + BA_{i-1} = \rho^{p^n} B^{p^{n(m-i+1)}}.$$

Changing the summation index from l to j, we have

$$\sum_{j=0}^{m-1} B^{p^{nj}} A_{i+j-1}^{p^{nj}} = \rho^{p^n} B^{p^{n(m-i+1)}}. \qquad (3)$$

Aside from the factor ρ^{p^n}, formula (3) is identical with the $(i-1)^{\text{st}}$ formula of the set (2). A condition for the invariance of the function Z is that the factor ρ satisfy the equation

$$\rho^{p^n} = \rho.$$

With this restriction upon ρ, all of the m formulæ (2) are consequences of a single one of them, say that given by $i = m$. We may thus enunciate the following

Theorem: *The totality of substitutions* (A) *for which*

$$\rho \equiv B^{-1} \sum_{j=0}^{m-1} B^{p^{nj}} A_j^{p^{nj}}$$

is a mark of the $GF\,[p^n]$ *form a group whose substitutions multiply the function* Z *by the parameter* ρ.

Note.—Since the function Z belongs to the $GF[p^n]$, the corresponding linear group is that subgroup of the general m-ary linear homogeneous group in the $GF[p^n]$ which leaves relatively invariant a certain linear function Z of the m variables.

3. By way of illustration of the general developments of §4, we consider the special case of the group of substitutions in the $GF[p^{3n}]$ on the variable X,

$$X' = A_1 X^{p^{2n}} + A_2 X^{p^n} + A_3 X,$$

which multiply by a parameter ρ the function

$$Y \equiv X X^{p^n} + X^{p^n} X^{p^{2n}} + X^{p^{2n}} X.$$

To form the transformed function Y', we note that

$$X' X'^{p^n} = A_3 A_1^{p^n} X^2 + (A_3^{p^n+1} + A_2 A_1^{p^n}) X^{p^n+1} + A_2 A_3^{p^n} X^{2p^n} \\ + (A_3 A_2^{p^n} + A_1^{p^n+1}) X^{p^{2n}+1} + (A_2^{p^n+1} + A_1 A_3^{p^n}) X^{p^{2n}+p^n} + A_1 A_2^{p^n} X^{2p^{2n}}.$$

Raising this equation to the powers p^n and p^{2n} and adding the three results, we find that the conditions for the identity

$$Y' \equiv X' X'^{p^n} + X'^{p^n} X'^{p^{2n}} + X'^{p^{2n}} X' = \rho Y$$

are the following six relations:

$$f \equiv A_3^{p^n+1} + A_2 A_1^{p^n} + A_3^{p^n} A_2^{p^{2n}} + A_1^{p^{2n}+p^n} + A_2^{p^{2n}+1} + A_1^{p^{2n}} A_3 = \rho, \qquad (4)$$

$$f^{p^n} = \rho, \qquad f^{p^{2n}} = \rho, \qquad (5)$$

$$A_3 A_1^{p^n} + A_1^{p^n} A_2^{p^{2n}} + A_2^{p^{2n}} A_3 = 0, \qquad (6)$$

together with (6) raised to the powers of p^n and p^{2n}.

Those substitutions in which the marks A_1, A_2, A_3 of the $GF[p^{3n}]$ satisfy the condition (6) *and give to the function f a value belonging to the $GF[p^n]$, form a group leaving Y invariant up to the factor f.*

4. Consider the substitutions (A) of the Betti-Mathieu Group which leave relatively invariant the function

$$Y_s \equiv \sum_{j=0}^{m-1} (BX)^{p^{nj}} (CX)^{p^{n(s+j)}},$$

where B, C and X belong to the $GF[p^{nm}]$ and s is any integer $< m$. We observe that

$$Y_s^{p^n} = Y_s,$$

so that Y_s belongs to the $GF[p^n]$. Applying to Y_s the substitution (A) and making use of formula (1), we find that

$$\left.\begin{aligned} Y'_s &\equiv \sum_{j=0}^{m-1} (BX')^{p^{nj}} (CX')^{p^{n(s+j)}} \\ &= \sum_{j=0}^{m-1} B^{p^{nj}} C^{p^{n(s+j)}} \sum_{i,\,l}^{1\,\ldots\,m} A_{i+j}^{p^{nj}} A_{l+s+j}^{p^{n(s+j)}} X^{p^{n(m-i)}} X^{p^{n(m-l)}} \end{aligned}\right\}$$

$$= \sum_{i=1}^{m} D_{ii} X^{2p^{n(m-i)}} + \sum_{i<l}^{i,\,l=1\,\ldots\,m} D_{il} X^{p^{n(m-i)}} X^{p^{n(m-l)}},$$

where we have used the abbreviations

$$D_{ii} \equiv \sum_{j=0}^{m-1} B^{p^{nj}} C^{p^{n(s+j)}} A_{i+j}^{p^{nj}} A_{i+s+j}^{p^{n(s+j)}},$$

$$D_{il} \equiv \sum_{j=0}^{m-1} B^{p^{nj}} C^{p^{n(s+j)}} \left(A_{i+j}^{p^{nj}} A_{l+s+j}^{p^{n(s+j)}} + A_{l+j}^{p^{nj}} A_{i+s+j}^{p^{n(s+j)}}\right).$$

The subscripts to D_{il}, like those to A_i, are to be taken modulo m.

4_1. Suppose first that s is neither 0 nor $m/2$. The powers of X in the terms of Y_s have, then, distinct exponents. We may write Y_s in the form

$$Y_s \equiv \sum_{i=1}^{m-s} (BX)^{p^{n(m-i-s)}} (CX)^{p^{n(m-i)}} + \sum_{i=1}^{s} (BX)^{p^{n(m-i)}} (CX)^{p^{n(s-i)}}.$$

The identity $Y'_s = \rho Y_s$, where ρ is a parameter, thus imposes upon the coefficients A_i the following conditions:

$$D_{ii} = 0, \qquad (i = 1, 2, \ldots, m) \tag{7}$$

$$D_{i\,i+s} = \rho B^{p^{n(m-i-s)}} C^{p^{n(m-i)}}, \qquad (i = 1, \ldots, m-s) \tag{8}$$

$$D_{i\,i+m-s} = \rho B^{p^{n(m-i)}} C^{p^{n(s-i)}}, \qquad (i = 1, \ldots, s) \tag{9}$$

$$D_{il} = 0. \qquad \begin{pmatrix} i, l = 1, \ldots, m;\ i < l \\ l \neq i,\ i+s, \text{ or } i+m-s \end{pmatrix} \tag{10}$$

We verify immediately that

$$D_{il}^{p^n} = D_{i-1\,l-1}. \tag{11}$$

The conditions (7) thus reduce to a single one, as $D_{11} = 0$. The conditions (8) are consequences of a single one, in view of (11), provided ρ satisfies the relation $\rho^{p^n} = \rho$. Similarly for the conditions (9). Further, we may verify that the ρ calculated from (8) equals $\rho^{p^{ns}}$ as calculated from (9). Hence (8) and (9) reduce to the single condition that the value for ρ shall belong to the $GF[p^n]$. Finally, the $\frac{1}{2}m(m-1)-m$ conditions (10) reduce to $\frac{1}{2}(m-3)$ or $\frac{1}{2}(m-2)$ according as m is odd or even. Indeed, by (11), we may retain only the conditions $D_{1l}=0$. From the symmetry of D_{il}, it equals D_{li}. Hence from $D_{1l}=0$ follows $D_{l1}=0$, and by (11), $D_{1\,m+s-l}=0$.

The two equivalent equations of one pair,

$$D_{1l}=0, \quad D_{1\,m+2-l}=0, \quad \begin{pmatrix} l=2,\ldots, m, \, l\neq 1+s, \\ l\neq 1+m-s \end{pmatrix} \tag{12}$$

are identical only when $l=\frac{1}{2}(m+2)$, i. e., when m is even. The two equations excluded in (12),

$$D_{1\,1+s}=0, \quad D_{1\,1+m-s}=0,$$

would have formed a pair of equivalent equations. There remain $\frac{1}{2}(m-3)$ pairs if m be odd and $\frac{1}{2}(m-4)$ pairs with an additional middle equation if m be even. We have proven the theorem:

For $s\neq 0$, $\neq \frac{m}{2}$, *the number of independent conditions upon the* m *coefficients* A_i *of a substitution* (A) *in order that it leave relatively invariant the function* Y_s *is at most*

$\frac{1}{2}(m+1)$ *for* m *odd,*
$\frac{1}{2}(m+2)$ *for* m *even.*

4_2. For $s=0$, we may give Y_s the form

$$Y_0\equiv\sum_{i=1}^{m}(BC)^{p^{n(m-i)}}X^{2p^{n(m-i)}}.$$

The conditions for the invariance of Y_0 are, therefore,

$$D_{ii}=\rho(BC)^{p^{n(m-i)}}, \qquad (i=1,2,\ldots,m) \tag{13}$$

$$D_{il}=0. \qquad (i,\, l=1,\ldots, m;\, i<l) \tag{14}$$

As before, we derive from (13) the condition $\rho^{p^n}=\rho$, in virtue of which the conditions (13) reduce to a single one. The conditions (14) reduce to $\frac{1}{2}(m-1)$ if m be odd, and to $\frac{1}{2}m$ if m be even. The above theorem, therefore, holds true if $s=0$.

4_3. If $s=m/2$, the terms in Y_s have in pairs like powers of X, viz., the jth and $s+j$th. The conditions become more complicated.

5. To the groups in §§3–4 there correspond certain linear homogeneous m-ary groups defined by single quadratic invariants. Indeed, if I be a root of a congruence of degree m belonging to and irreducible in the $GF[p^n]$, we may set

$$X=\sum_{j=0}^{m-1}\xi_j I^j,\quad B=\sum_{j=0}^{m-1}\beta_j I^j,\quad C=\sum_{j=0}^{m-1}\gamma_j I^j,$$

where ξ_j, β_j and γ_j are marks of the $GF[p^n]$. Then, for example, $(BX)^{p^n}$ becomes a *linear* function of $\xi_0, \xi_1, \ldots, \xi_{m-1}$, since

$$\xi_j^{p^n}=\xi_j.$$

Hence Y_s becomes a quadratic function of the ξ's. As noted above, Y_s belongs to the $GF[p^n]$. Hence our quadratic function of the ξ's has for coefficients certain marks of the $GF[p^n]$. The corresponding m-ary linear group is, therefore, defined by a single quadratic invariant. The structure of all such groups has been fully determined by the writer in the American Journal of Mathematics for July, 1899.

THE UNIVERSITY OF TEXAS, *July* 14, 1899.

On the Excess of the Number of Combinations in a Set which have an Even Number of Inversions over those which have an Odd Number.

By W. H. Metzler, Ph. D.

1. If we are given any combination of n numbers m at a time, the combination of the remaining $n-m$ numbers is said to be the complementary with respect to n of the given combination.

Let it be understood (unless otherwise expressed) that the numbers in any combination are arranged in their natural order (order of magnitude). Let $(n|\underset{1}{m})$, $(n|\underset{2}{m})\ldots\ldots(n|\underset{\mu}{m})$ denote the $\dfrac{n(n-1)\ldots\ldots(n-m+1)}{m!}=n_m=\mu$ combinations of the numbers 1, 2, 3 … n taken m at a time, and let $(\overline{n}|\underset{1}{m})\ldots(\overline{n}|\underset{\mu}{m})$ denote their complementaries. Let $(n|\underset{\alpha}{m}|\underset{1}{l})$, $(n|\underset{\alpha}{m}|\underset{2}{l})\ldots\ldots(n|\underset{\alpha}{m}|\underset{\lambda}{l})$ denote the $n_l=\lambda$ combinations of the numbers in the combination $(n|\underset{\alpha}{m})$ taken l at a time, and let $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})$ denote the combination which is the complementary with respect to m of the combination $(n|\underset{\alpha}{m}|\underset{\beta}{l})$, i. e., the combination of the $m-l$ numbers remaining after the numbers in the combination $(n|\underset{\alpha}{m}|\underset{\beta}{l})$ are taken out of the combination $(n|\underset{\alpha}{m})$. For present purposes let $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l})$ denote the combination made up of the numbers in $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})$ followed by the numbers in $(n|\underset{\alpha}{m}|\underset{\beta}{l})$. In contrast with this I have used elsewhere* $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l})$ to denote the combination of the numbers in the two combinations $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})$ and $(n|\underset{\alpha}{m}|\underset{\beta}{l})$ arranged in their natural order.

---

* Am. Jour. Math., vol. XX, No. 3.

Let any combination having an odd number of inversions from the natural order be affected with the negative sign.

2. If k denote the number of inversions in $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l})$, then it may be easily proven that the number of inversions in $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l})$ is $l(m-l)-k$.

Therefore

$$\begin{aligned}(n|\underset{\alpha}{m}|\underset{\beta}{l})(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l}) &= (-1)^{l(m-l)-2k}(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l})\\ &= (-1)^{l(m-l)}(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l}).\end{aligned}$$

3. The combination $(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l}) = \pm(n|\underset{\alpha}{m})$ according as the number of inversions is even or odd.

Let

$$(n|\underset{\alpha}{\overline{m}}|\underset{1}{l})(n|\underset{\alpha}{m}|\underset{1}{l}) + (n|\underset{\alpha}{\overline{m}}|\underset{2}{l})(n|\underset{\alpha}{m}|\underset{2}{l}) + \dots + (n|\underset{\alpha}{\overline{m}}|\underset{\lambda}{l})(n|\underset{\alpha}{m}|\underset{\lambda}{l}) = \phi(m,l)\,.\,(n|\underset{\alpha}{m}), \tag{1}$$

then will

$$(n|\underset{\alpha}{m}|\underset{1}{l})(n|\underset{\alpha}{\overline{m}}|\underset{1}{l}) + (n|\underset{\alpha}{m}|\underset{2}{l})(n|\underset{\alpha}{\overline{m}}|\underset{2}{l}) + \dots + (n|\underset{\alpha}{m}|\underset{\lambda}{l})(n|\underset{\alpha}{\overline{m}}|\underset{\lambda}{l}) = \phi(m, m-l)\,.\,(n|\underset{\alpha}{m}).$$

But $$(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l})(n|\underset{\alpha}{m}|\underset{\beta}{l}) = (-1)^{l(m-l)}(n|\underset{\alpha}{m}|\underset{\beta}{l})(n|\underset{\alpha}{\overline{m}}|\underset{\beta}{l}), \qquad \text{(art. 2)}$$

$$\therefore\ \phi(m,l) = (-1)^{l(m-l)}\phi(m, m-l).$$

If $l=1$, the signs of the left-hand member of equation (1) are evidently alternately positive and negative, therefore,

$$\phi(m,1) = 1 \text{ or } 0$$

according as m is odd or even.

It is also apparent that

$$\phi(m,m) = 1.$$

4. The set of combinations

$$(n|\underset{\alpha}{\overline{m}}|\underset{1}{l})(n|\underset{\alpha}{m}|\underset{1}{l}),\ (n|\underset{\alpha}{\overline{m}}|\underset{2}{l})(n|\underset{\alpha}{m}|\underset{2}{l}),\ \dots,\ (n|\underset{\alpha}{\overline{m}}|\underset{\lambda}{l})(n|\underset{\alpha}{m}|\underset{\lambda}{l})$$

may be divided up into groups as follows:

The first group containing the first $(m-1)_{l-1}$ combinations,
" second " " " next $(m-2)_{l-1}$, "
....
" $(m-l+1)^{\text{st}}$ " " " last $(l-1)_{l-1}=1$ "

The first number in the second part of each combination of the r^{th} group is the same, and is the r^{th} of the selection of m numbers, i. e., the r^{th} of the numbers in $(n \underset{a}{|} m)$. The first $r-1$ numbers in the first part of each combination of the r^{th} group are the same and are the first $r-1$ of the numbers in $(n \underset{a}{|} m)$. It follows from this that the signs of the combinations of the r^{th} group are the same as or the opposite to (according as $m-l-r+1$ is even or odd, there being $m-l-r+1$ numbers in the first part greater than the first number in the second part) the signs of the corresponding members of the set obtained by striking out the $r-1$ numbers common to the first part and the one number common to the second part of each combination of the group.

We have, therefore

$$\begin{aligned}\phi(m, l) = {} & \phi(l-1, l-1) - \phi(l, l-1)\, \phi(l+1, l-1) \dots \\ & + (-1)^{m-l-r+1}\phi(m-r, l-1) + \dots + (-1)^{m-l}\phi(m-1, l-1), \qquad (2)\end{aligned}$$

a reduction formula for $\phi(m, l)$.

As an immediate consequence of equation (2), we have

$$\begin{aligned}\phi(m, l) &= \phi(m-r, l) + (-1)^{m-r-l+1}\phi(m-r, l-1) \\ &\qquad + (-1)^{m-l}\phi(m-1, l-1) \\ &= \phi(m-1, l) + (-1)^{m-l}\phi(m-1, l-1). \qquad (3)\end{aligned}$$

5. By successive applications of equation (2), we have

$$\begin{aligned}\phi(2m, 2l+1) &= \phi(2l, 2l) - \phi(2l+1, 2l) + \dots - \phi(2m-1l\ 2l) \\ &= \phi(2l-1, 2l-1) \\ &\quad - \phi(2l-1, 2l-1) + \phi(2l, 2l-1) \\ &\quad + \phi(2l-1, 2l-1) - \phi(2l, 2l-1) + \phi(2l+1, 2l-l) \\ &\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\quad - \phi(2l-1, 2l-1) + \dots \qquad + \phi(2m-2, 2l-1) \\ &= \phi(2l, 2l-1) + \phi(2l+2, 2l-1) + \dots \\ &\qquad + \phi(2m-2, 2l-1). \qquad (4)\end{aligned}$$

Put $l=1$; then

$$\begin{aligned}\phi(2m, 3) &= \phi(2, 1)+\phi(4, 1)+\dots+\phi(2m-2, 1)\\ &= 0. \qquad \text{(art. 3)}.\end{aligned}$$

Put $l=2$, then

$$\begin{aligned}\phi(2m, 5) &= \phi(4, 3)+\phi(6, 3)+\dots+\phi(2m-2, 3)\\ &= 0.\end{aligned}$$

In this way it may be shown that

$$\phi(2m, 2l+1)=0, \qquad (l=1, 2 \dots \overline{m-1}). \tag{5}$$

6. From equations (3) and (5), we have

$$\begin{aligned}\phi(2m+1, 2l) &= \phi(2m, 2l)-\phi(m, 2l-1)\\ &= \phi(2m, 2l),\\ \phi(2m+1, 2l+1) &= \phi(2m, 2l+1)+\phi(2m, 2l)\\ &= \phi(2m, 2l).\end{aligned}$$

$$\therefore\ \phi(2m+1, 2l+1)=\phi(2m+1, 2l)=\phi(2m, 2l). \tag{6}$$

It follows from this and art. 3 that

$$\phi(m, l)=\phi(m, m-l).$$

If $l=m$, then

$$\phi(m, m)=\phi(m, 0)=1.$$

7. From equations (2), (5) and (6), we have

$$\begin{aligned}\phi(2m, 2l) &= \phi(2m-1, 2l-1)+\phi(2m-3, 2l-1)+\dots\\ &\qquad +\phi(2l-1, 2l-1)\\ &= \phi(2m-1, 2l-2)+\phi(2m-4, 2l-2)+\dots\\ &\qquad +\phi(2l-2, 2l-2).\end{aligned} \tag{7}$$

These properties at once suggest that

$$\phi(2m, 2l)=m_l,$$

and it may be easily proved that this is true.

If, in equation (7), we put—

1st. $l=1$, then

$$\begin{aligned}\phi(2m, 2) &= \phi(2m-2, 0)+\phi(2m-4, 0)+\dots+\phi(0, 0)\\ &= m \text{ or } m_1;\end{aligned}$$

2nd. $l = 2$, then

$$\begin{aligned}\phi(2m, 4) &= \phi(2m-2, 2) + \phi(2m-4, 2) + \dots + \phi(2, 2)\\ &= (m-1)_1 + (m-2)_1 + \dots + 1_1\\ &= m_2;\end{aligned}$$

3rd. $l = 3$, then

$$\begin{aligned}\phi(2m, 6) &= \phi(2m-2, 4) + \phi(2m-4, 4) + \dots + \phi(4, 4)\\ &= (m-1)_2 + (m-2)_2 + \dots + 2_2\\ &= m_3.\end{aligned}$$

In this way we see from equation (7) itself, that if it is true for any value of l, it is true for a value one greater and, therefore, true for all values. Hence

$$\phi(2m, 2l) = m_l; \qquad (l = 1, 2 \dots m)$$

SYRACUSE UNIVERSITY, *February* 10, 1898.

On Lie's Theory of Continuous Groups.

BY E. W. RETTGER.

An r-parameter group generated by the infinitesimal transformations $X_1, X_2, \ldots, X_r$ is continuous if each transformation of the group, without exception, belongs to a one-parameter group whose equations, at least in the neighborhood of the identical transformations, are

$$x_i' = x_i + \sum_1^r l_j X_j x_i + \frac{1}{1 \,.\, 2} \sum_1^r \sum_1^r l_j l_k X_j X_k x_i + \ldots, \qquad (1)$$

$$(i = 1, 2, \ldots, n)$$

that is to say, if each transformation of the group can be generated by an infinitesimal transformation of the group.

In his "Continuerliche Gruppen," Lie demonstrated that every transformation of the general projective group can be generated by an infinitesimal transformation of this group. Whence it follows that this group is continuous.* But Professor Study's important discovery† that not every transformation of the special linear homogeneous group can be generated by an infinitesimal transformation of the special linear homogeneous group, shows that not every subgroup of the general projective group is continuous, as was apparently assumed by Lie.

In this paper I shall term a transformation of an r-parameter group G_r that cannot be generated by the repetition of an infinitesimal transformation of this group a *singular transformation* of the group G_r. Study pointed out that the special linear homogeneous group in two variables contained singular transformations. Subsequently Professor Taber established the existence of singular transformations in the group of orthogonal substitutions in n variables, for $n \geq 4$, and also in the group of linear automorphic transformations of an alternate bilinear form, and the group of linear automorphic transformations of a general bilinear form.*

---

* Lie, "Continuerliche Gruppen," p. 45.

† "Leipzige Berichte," 1892.

In what follows I shall investigate the two- and three-parameter subgroups of the general projective group in two variables, and of the general linear homogeneous group in three variables, enumerated by Lie on pages 288 and 519 respectively, of his "Continuerliche Gruppen;" and I propose to show that singular transformations occur among the transformations of many of these subgroups.

Let $X_1, X_2, \ldots., X_r$ be the r infinitesimal transformations that generate a given r-parameter group G_r. I shall denote by T_a the transformation defined by the equations

$$x_i' = x_i + \sum_{j=1}^{r} a_j X_j x_i + \frac{1}{1\,.\,2} \sum_{j=1}^{r} \sum_{k=1}^{r} a_j a_k X_j X_k x_i + \ldots.,$$
$$(i = 1, 2, \ldots., n)$$

and, therefore, generated by the infinitesimal transformation whose symbol is

$$a_1 X_1 + \ldots. + a_r X_r.$$

Similarly, T_b and T_c will denote the transformations defined respectively by

$$x_i' = x_i + \sum_{j=1}^{r} b_j X_j x_i \frac{1}{1\,.\,2} \sum_{j=1}^{r} \sum_{k=1}^{r} b_j b_k X_j X_k x_i + \ldots.,$$
$$(i = 1, 2, \ldots., n)$$

and

$$x_i' = x_i + \sum_{j=1}^{r} c_j X_j x_i + \frac{1}{1\,.\,2} \sum_{j=1}^{r} \sum_{k=1}^{r} c_j c_k X_j X_k x_i + \ldots.,$$
$$(i = 1, 2, \ldots., n)$$

and, therefore, generated respectively by the infinitesimal transformations

$$b_1 X_1 + \ldots. + b_r X_r,$$

and

$$c_1 X_1 + \ldots. + c_r X_r.$$

Furthermore, $T_b T_a$ will denote the composition of T_a and T_b in the order named, that is, the transformation resulting from the successive application to the manifold $(x_1, x_2, \ldots, x_n)$ of the transformations T_a and T_b in the order named.[*]

---

* Am. Journ. Math., vol. XVI, p. 180; Bul. N. Y. Math. Soc., July, 1894; Math. Ann., vol. XLVI, p. 561; Math. Review, vol. I, p. 154.

In general, whatever be the values of the a's and b's, we can put

$$T_b\,T_a = T_c;$$

where
$$c_k = \lambda_k(a_1 \ldots a_r\; b_1 \ldots b_r); \qquad (k = 1, 2, \ldots, r) \tag{2}$$

that is to say, the composition of two transformations T_a and T_b generated respectively by the infinitesimal transformations $aX_1 + \ldots + a_r X_r$, and $b_1 X_1 + \ldots + b_r X_r$, results, in general, in a transformation, T_c, which can also be generated by an infinitesimal transformation, viz.:

$$c_1 X_1 + \ldots + c_r X_r.$$

Now, a transformation of G_r can often be generated by more than one infinitesimal transformation of G_r. In this case, equations (2) give more than one set of values for the c's that correspond to the particular set of values assigned to the a's and b's; that is to say, *the c's are multivalued functions of the a's and b's.* Let $T_b\,T_a$ be finite for $a_j = \bar{a}_j$ and $b_j = \bar{b}_j$, $(j = 1, 2 \ldots r)$. It may be, for this system of values of the a's and b's, that—

1st. Every branch of each function $\lambda_1 \ldots \lambda_r$ is finite and determinate;

2nd. One or more, but not all, of the branches of one (or more) of the functions $\lambda_1 \ldots \lambda_r$ is infinite or indeterminate;

3rd. Each branch of one (or more) of the functions $\lambda_1 \ldots \lambda_r$ is infinite or indeterminate.

Only in the last case do equations (2) fail to give us the parameters of an infinitesimal transformation which will generate $T_b\,T_a$. In this case, there is no infinitesimal transformation of G_r that will generate $T_b\,T_a$; that is to say, $T_b\,T_a$ is a *singular transformation.*

To ascertain whether G_r contains singular transformations, we have then (knowing $\lambda_1 \ldots \lambda_r$) to ascertain whether one (or more) sets of values of the a's and b's will make each branch of one (or more) of the functions $\lambda_1 \ldots \lambda_r$

---

[*] Lie denotes by $T_b\,T_a$, the transformation resulting from the application to the manifold $(x_1, x_2, \ldots, x_n)$, first of the transformation T_b, and then of the transformation T_a.

infinite or indeterminate. If no such system of values of the a's and b's can be found, G_r contains no singular transformation. If, on the other hand, one or more such systems of values of the a's and b's can be found, then the group G_r contains one or more singular transformations.

To determine $\lambda_1, \lambda_2, \ldots, \lambda_r$, let the summation of the infinite series

$$x_i' = x_i + \sum_1^r l_j X_j x_i + \frac{1}{1 \cdot 2} \sum_1^r \sum_1^r l_j l_k X_j X_k x_i + \ldots \qquad (1)$$

$$(i = 1, 2, \ldots, n)$$

give

$$x_i' = \phi_i(x_1 \ldots x_n, l_1 \ldots l_r); \qquad (i = 1, 2, \ldots, n)$$

or, in the abbreviated notation,

$$x_i' = \phi_i(x, l). \qquad (i = 1, 2, \ldots, n) \qquad (3)$$

Then T_a and T_b are defined respectively by the equations

$$x_i' = \phi_i(x, a)$$

and

$$x_i'' = \phi_i(x', b). \qquad (i = 1, 2, \ldots, n)$$

Eliminating $x_1' \ldots x_n'$ from the last two systems of equations, we obtain, as a consequence of the fundamental property of a group,

$$x_i'' = \phi_i(x, \lambda(ab)). \qquad (i = 1, 2, \ldots, n)$$

If the transformation defined by this system of equations is identical with the transformation T_c, this system of equations is equivalent to the system

$$x_i'' = \phi_i(x, c); \qquad (i = 1, 2, \ldots, n)$$

and, from this identity, we derive the system of equations

$$c_k = \lambda_k(a, b). \qquad (k = 1, 2, \ldots, r) \qquad (2)$$

I shall apply the foregoing to several subgroups of the general projective group of the plane.

The subgroup, q, yq, xp.* For this group $\sum_1^r l_j X_j \equiv l_1 q + l_2 yq + l_3 xp$, and equations (1) become

---

* Lie employs p and q to denote respectively $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$x' = x + l_3 x + \tfrac{1}{2} l_3^2 x + \ldots . = e^{l_3} x,$$
$$y' = y + l_1 + l_2 y + \frac{l_2 l_1 + l_2^2 y}{2} + \ldots . = e^{l_2} y + \frac{l_1}{l_2}(e^{l_2} - 1).$$

Consequently, T_a and T_b are defined respectively by the equations

$$\left.\begin{aligned} x' &= e^{a_3} x, \\ y' &= e^{a_2} y + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}\right\} \tag{A}$$

and

$$\left.\begin{aligned} x'' &= e^{b_3} x', \\ y'' &= e^{b_2} y' + \frac{b_1}{b_2}(e^{b_2} - 1). \end{aligned}\right\} \tag{B}$$

Eliminating x' and y' between equations (A) and (B), we obtain, as the system of equations defining $T_b\, T_a$,

$$\left.\begin{aligned} x'' &= e^{a_3 + b_3} x, \\ y'' &= e^{a_2 + b_2} y + N, \end{aligned}\right\} \tag{C}$$

where N denotes

$$e^{b_2} \frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1).$$

Moreover, T_c is defined by the system of equations

$$\left.\begin{aligned} x'' &= e^{c_3} x, \\ y'' &= e^{c_2} y + \frac{c_1}{c_2}(e^{c_2} - 1). \end{aligned}\right\} \tag{D}$$

Therefore, if $T_b\, T_a = T_c$, that is, if the system of equations (C) is equivalent to the system of equations (D),

$$e^{c_3} = e^{a_3 + b_3},$$
$$e^{c_2} = e^{a_2 + b_2},$$
$$\frac{c_1}{c_2}(e^{c_2} - 1) = N.$$

Consequently, equations (2) become

$$\left.\begin{aligned} c_3 &= a_3 + b_3 + 2m\pi i, \\ c_2 &= a_2 + b_2 + 2n\pi i, \\ c_1 &= \frac{N(a_2 + b_2 + 2n\pi i)}{e^{a_2 + b_2} - 1}, \end{aligned}\right\} \tag{E}$$

where m and n are arbitrary integers or are equal to zero.

From equations (E), it follows that c_3 and c_2 are finite for all sets of finite values of the a's and b's. Moreover, c_1 is finite for finite values of the a's and b's, unless $a_2 + b_2$ is an even multiple, $2n'$, of πi. But in this case, if we put $n = -n'$, we have $c_1 = N$ which is certainly finite. Therefore, c_1, c_2, and c_3 can all be taken finite for finite values of the a's and b's. Whence, it follows that this group does not contain singular transformations. We have here an illustration of the 2nd case mentioned on page 62.

The subgroup, q, $yq + p$. For this group $\sum_1^r l_j X_j \equiv l_1 q + l_2(yq + p)$; and equations (1) become

$$x' = x + l_2,$$
$$y' = e^{l_2} y + \frac{l_1}{l_2}(e^{l_2} - 1).$$

Consequently, T_a and T_b are defined respectively by

$$\left.\begin{aligned} x' &= x + a_2, \\ y' &= e^{a_2} y + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}\right\} \tag{A}$$

and

$$\left.\begin{aligned} x'' &= x' + b_2, \\ y'' &= e^{b_2} y' + \frac{b_1}{b_2}(e^{b_2} - 1); \end{aligned}\right\} \tag{B}$$

and $T_b\, T_a$ by

$$\left.\begin{aligned} x'' &= x + a_2 + b_2, \\ y'' &= e^{a_2 + b_2} y + N, \end{aligned}\right\} \tag{C}$$

where N denotes $e^{b_2} \frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1)$. Moreover, T_c is defined by

$$\left.\begin{aligned} x'' &= x + c_2, \\ y'' &= e^{c_2} y + \frac{c_1}{c_2}(e^{c_2} - 1). \end{aligned}\right\} \tag{D}$$

Therefore, if $T_b\, T_a = T_c$,

$$c_2 = a_2 + b_2,$$

$$\frac{c_1}{c_2}(e^{c_2} - 1) = N.$$

Consequently, equations (2) become

$$\left.\begin{aligned} c_2 &= a_2 + b_2, \\ c_1 &= \frac{N(a_2 + b_2)}{e^{a_2 + b_2} - 1}. \end{aligned}\right\} \qquad \text{(E)}$$

From the first of equations (E), it follows that c_2 is finite for all finite values of the a's and b's. If $a_2 + b_2 = 0$, it follows from the last of these equations that c_1, which in this case equals N, is finite; but if $a_2 + b_2 = 2m\pi i$ for some integer $m \neq 0$, c_1 is infinite, provided $N \neq 0$. Now, we can so choose the a's and b's that $a_2 + b_2 = 2m\pi i \neq 0$, and at the same time that $N \neq 0$. For let $a_2 + b_2 = 2m\pi i \neq 0$, then

$$N = \left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right)(1 - e^{-a_2}).$$

By properly choosing the a's and b's, subject to the negative condition that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ and that a_2 is not an integer-multiple of $2\pi i$, N may be taken arbitrarily. We may, therefore, assume $N \neq 0$. We thus obtain a transformation $T_b\, T_a$ of our group,

$$\begin{aligned} x'' &= x + 2m\pi i, \qquad && (m \text{ an integer} \neq 0) \\ y'' &= y + N, && (N \neq 0) \end{aligned}$$

which cannot be generated by an infinitesimal transformation of our group. Consequently, the given group contains singular transformations.

I shall now develop a few theorems, more or less evident, that will be made use of in this paper. Let the two transformations T_a and T_b of G_r, generated respectively by $\sum_1^r a_s X_s$ and $\sum_1^r b_s X_s$, be applied successively to the manifold $(x_1 \ldots . x_n)$. We have then the simultaneous systems of equations

$$x_i' = x_i + \sum_1^r a_s X_s\, x_i + \tfrac{1}{2}\Big(\sum_1^r a_s X_s\Big)^2 x_i + \ldots . \qquad (4)$$

$$(i = 1, 2, \ldots . , n)$$

and

$$x_i'' = x_i' + \sum_1^r b_s X_s'\, x_i' + \tfrac{1}{2}\Big(\sum_1^r b_s X_s'\Big)^2 x_i' + \ldots . , \qquad (5)$$

$$(i = 1, 2, \ldots . , n)$$

where X_s' denotes the result of substituting in X_s the x''s for the x's, and $\left(\sum_1^r a_s X_s\right)^2 x_i$, etc., denote respectively $\sum_1^r a_s X_s \left(\sum_1^r a_s X_s x_i\right)$, etc.

On eliminating $x_1', x_2', \ldots, x_n'$ from the systems of equations (4) and (5), we obtain the equations*

$$x_i'' = x_i + \left(\sum_1^r a_s X_s + \sum_1^r b_s X_s\right) x_i + \frac{1}{1\,.\,2}\left[\left(\sum_1^r a_s X_s\right)^2 + 2\sum_1^r b_s X_s \,.\, \sum_1^r a_s X_s + \left(\sum_1^r b_s X_s\right)^2\right] x_i + \ldots\ldots, \quad (6)$$

$$(i = 1, 2, \ldots\ldots, n)$$

which define the transformation $T_b\, T_a$.

Let now the infinitesimal transformations $\sum_1^r a_s X_s$ and $\sum_1^r b_s X_s$ be commutative; that is, let $\left(\sum_1^r a_s X_s, \ \sum_1^r b_s X_s\right) = 0$; or, what is the same thing, let $\sum_1^r a_s X_s \,.\, \sum_1^r b_s X_s = \sum_1^r b_s X_s \,.\, \sum_1^r a_s X_s$. Then $\left(\sum_1^r a_s X_s\right)^2 + 2\sum_1^r b_s X_s \,.\, \sum_1^r a_s X_s + \left(\sum_1^r b_s X_s\right)^2 = \left(\sum_1^r a_s X_s\right)^2 + \sum_1^r a_s X_s \,.\, \sum_1^r b_s X_s + \sum_1^r b_s X_s \,.\, \sum_1^r a_s X_s + \left(\sum_1^r b_s X_s\right)^2$ $= \left(\sum_1^r a_s X_s + \sum_1^r b_s X_s\right) . \left(\sum_1^r a_s X_s + \sum_1^r b_s X_s\right) = \left(\sum_1^r a_s X_s + \sum_1^r b_s X_s\right)^2$ $= \left(\sum_1^r \overline{a_s + b_s}\, X_s\right)^2$. Similarly, $\left(\sum_1^r a_s X_s\right)^3 + 3\left(\sum_1^r b_s X_s\right) . \left(\sum_1^r a_s X_s\right)^2$ $+ 3\left(\sum_1^r b_s X_s\right)^2 . \sum_1^r a_s X_s + \left(\sum_1^r b_s X_s\right)^3 = \left(\sum_1^r a_s X_s + \sum_1^r b_s X_s\right)^3 = \left(\sum_1^r \overline{a_s + b_s}\, X_s\right)^3$, etc. Consequently, if $\left(\sum_1^r a_s X_s, \ \sum_1^r b_s X_s\right) = 0$, equations (6) may be written

$$x_i'' = x_i + \sum_1^r \overline{a_s + b_s}\, X_s\, x_i + \frac{1}{1\,.\,2}\left(\sum_1^r \overline{a_s + b_s}\, X_s\right)^2 x_i + \frac{1}{1\,.\,2\,.\,3}\left(\sum_1^r \overline{a_s + b_s}\, X_s\right)^3 x_i + \ldots;$$

$$(i = 1, 2, \ldots\ldots, n)$$

---

* Lie, "Continuerliche Gruppen," p. 487.

and hence, in this case, $T_b T_a = T_a T_b$.* If, moreover, $T_b T_a = T_c$, where T_c is defined by

$$x_i'' = x_i + \sum_1^r c_s X_s x_i + \frac{1}{1\,.\,2}\Big(\sum_1^r c_s X_s\Big)^2 x_i + \frac{1}{1\,.\,2\,.\,3}\Big(\sum_1^r c_s X_s\Big)^3 x_i + \ldots\ldots;$$
$$(i = 1, 2, \ldots\ldots, n)$$

we may put $c_k = a_k + b_k$ for $k = 1, 2, \ldots\ldots, r$; and, therefore, the c's are finite for finite values of the a's and b's. We have thus the following theorem:

THEOREM I.—*If the two infinitesimal transformations* $\sum_1^r a_s X_s$ *and* $\sum_1^r b_s X_s$ *which generate respectively the finite transformations* T_a *and* T_b *of* G_r *are commutative, or what is the same thing, if* $\Big(\sum_1^r a_s X_s,\ \sum_1^r b_s X_s\Big) = 0$, *then the transformation* $T_b T_a$ *resulting from their composition is equivalent to a non-singular transformation* T_c *generated by the infinitesimal transformation* $\sum_1^r c_s X_s$, *where*

$$c_k = a_k + b_k. \qquad (k = 1, 2, \ldots\ldots, n)$$

If, now, $(X_j, X_k) = 0$ for $j, k = 1, 2, \ldots\ldots, r$, then for *every* system of values of the a's and b's $\Big(\sum_1^r a_s X_s,\ \sum_1^r b_s X_s\Big) = 0$, and, therefore, by theorem I, $T_b T_a$ (which is now equivalent to $T_a T_b$ for every system of values of the a's and b's) is non-singular. Conversely, if $T_b T_a = T_a T_b$ for *every* system of values of the a's and b's, then $(X_j, X_k) = 0$ for $j, k = 1, 2, \ldots\ldots, r$; and, consequently, $T_b T_a$ is non-singular. Whence we derive—

THEOREM II.—*If the transformations of* G_r *are all commutative, or, what is the same thing, if* $(X_j, X_k) = 0$ *for* $j, k = 1, 2, \ldots\ldots, r$, *no transformation of* G_r *is singular.*

As a particular case, we have, since the transformations of a one-parameter group G_1 are commutative, *a one-parameter group,* G_1, *contains no singular transformations.*†

---

* Lie, "Continuerliche Gruppen," p. 487.

† That is to say, the composition of transformations generated by the same infinitesimal transformation can always be generated by this infinitesimal transformation.

Groups whose transformations are commutative need not, therefore, be considered in our investigation.

Let us next assume that the symbols of infinitesimal transformations of our group are not in general commutative, but that $(X_j, X_k) = 0$ for $k = 1, 2, \dots, r$. We shall then have $\left(X_j, \sum_1^r a_s X_s\right) = 0$ for every system of values of the a's; and, consequently, by theorem I, each transformation generated by X_j will be commutative with every transformation of our group.

Let, now, S_a denote a finite transformation generated by X_j, that is, let S_a be defined by the equations

$$x_i' = x_i + a_j X_j x_i + \frac{1}{1.2}(a_j X_j)^2 x_i + \frac{1}{1.2.3}(a_j X_j)^3 x_i + \dots .$$
$$(i = 1, 2, \dots, n).$$

Further, let T_a' and T_b' denote two finite transformations generated respectively by the infinitesimal transformations

$$a_1 X_1 + \dots a_{j-1} X_{j-1} + a_{j+1} X_{j+1} + \dots + a_r X_r$$

and $$b_1 X_1 + \dots b_{j-1} X_{j-1} + b_{j+1} X_{j+1} + \dots + b_r X_r.$$

If, now, T_a denotes the transformation generated by $a_1 X_1 + \dots a_{j-1} X_{j-1} + a_j X_j + a_{j+1} X_{j+1} + \dots a_r X_r$, then, by theorem I,

$$T_a = S_{a_j} T_a' = T_a' S_{a_j}.$$

Similarly, $$T_b = S_{b_j} T_b' = T_b' S_{b_j}.$$

Moreover, $$T_b T_a = S_{a_j} S_{b_j} T_b' T_a' = S_{a_j + b_j} T_b' T_a'.$$

Furthermore, from theorem I, it follows that if $T_b' T_a'$ is non-singular, $S_{a_j + b_j} T_a' T_a'$ is non-singular. For $S_{a_j + b_j}$ is generated by the infinitesimal transformation $(a_j + b_j) X_j$, and is, therefore, commutative with every transformation of our group. If, then, $T_b' T_a'$ is generated by the infinitesimal transformation $c_1^{(1)} X_1 + \dots + c_j^{(1)} X_j + \dots + c_r^{(1)} X_r$, $S_{a_j + b_j} T_b' T_a'$, by theorem I is generated by the infinitesimal transformation $c_1^{(1)} X_1 + \dots (c_j^{(1)} + a_j + b_j) X_j + \dots + c_r^{(1)} X_r$. Consequently, in order that $T_b T_a$ shall be singular, $T_b' T_a'$ must be singular, but $T_b' T_a'$ is independent of a_j and b_j. Whence we obtain

THEOREM III.—*If* T_a *and* T_b, *generated respectively by* $\sum_1^r a_s X_s$ *and* $\sum_1^r b_s X_s$, *are two transformations of the group* $X_1, \ldots X_j, \ldots X_r$; *and if* X_j *is commutative with each symbol of infinitesimal transformation, that is, if* $(X_j, X_k) = 0$, $(k = 1, 2, \ldots, r)$, *then the singularity of the transformation* $T_b T_a$ *is independent of the particular values assigned to* a_j *and* b_j.

We may, therefore, let $a_j = b_j = 0$ without affecting our results.

I find it convenient to employ the following notation in dealing with linear homogeneous groups. The transformation defined by the system of equations

$$x_i' = a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n$$
$$(i = 1, 2, \ldots, n)$$

may be denoted by T_a. The identical transformation defined by

$$x_i' = x_i, \qquad (i = 1, 2, \ldots n)$$

may be denoted by I. We have $IT = TI$. Further, the transformation defined by

$$x_i' = \sigma x_i, \qquad (i = 1, 2, \ldots, n)$$

where σ is independent of the x's, may be denoted by σI, or simply by σ; and the composition of T and σ, defined by the equations

$$x_i' = \sigma (a_{i1} x_1 + a_{i2} x_2 + \ldots a_{in} x_n),$$
$$(i = 1, 2, \ldots, n)$$

may be denoted by σT. We evidently have

$$\sigma T = T \sigma.$$

There exists a well-known relation between the general projective group of the plane (x, y) and the general linear homogeneous group in three variables, x_1, x_2, x_3.* The nature of this relation is made evident if we consider x_1, x_2, x_3 as homogeneous coordinates of the plane. Then the transformation T of the general linear homogeneous group determines a corresponding projective transformation $\mathbf{T}$ of the plane. To show this, let T be defined by the three equations

$$\left.\begin{aligned} x_1' &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3, \\ x_2' &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3, \\ x_3' &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3. \end{aligned}\right\} \qquad \text{(A)}$$

---

* Lie, "Continuerliche Gruppen." p. 500.

If, now, x_1', x_2', x_3' and x_1, x_2, x_3 be considered as homogeneous coordinates, these equations are equivalent to the two following equations in Cartesian coordinates:

$$\left.\begin{aligned} x' &= \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \\ y' &= \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}, \end{aligned}\right\} \qquad \text{(B)}$$

which define a projective transformation $\mathbf{T}$ of the plane (x, y). In the determination of this projective transformation $\mathbf{T}$, we are concerned only with the ratios of the a's in the equations defining T. Consequently, the transformations T and σT of the general linear homogeneous group, for every value of σ other than zero, correspond to the same projective transformation $\mathbf{T}$ of the plane. In other words:* To every transformation $\mathbf{T}$ of the general projective group in two variables (x, y) is associated ∞^1 transformations σT of the general linear homogeneous group in three variables (x_1, x_2, x_3); and, conversely, to ∞^1 transformations σT of the general linear homogeneous group in three variables is associated a single transformation $\mathbf{T}$ of the general projective group in two variables.

Further, to every subgroup Γ_r of the general projective group in two variables is associated two or more subgroups G_r, G_r' .... of the general linear homogeneous group in three variables, while to every subgroup G_r of the general linear homogeneous group in three variables is associated but *one* subgroup Γ_r of the general projective group in two variables. If the subgroup G_r of the general linear homogeneous group does not contain singular transformations, the corresponding subgroup Γ_r of the general projective group cannot contain singular transformations. On the other hand, if G_r contains a singular transformation T, it does not necessarily follow that the corresponding transformation $\mathbf{T}$ of Γ_r is singular. For $\mathbf{T}$ is associated with ∞^1 transformations σT; and, although T may be a singular transformation of G_r, σT, which is a transformation of at least one of the subgroups G_r' .... associated with Γ_r, may, for some value of σ, be non-singular: in which case $\mathbf{T}$ is also non-singular. If, however, it is impossible to find a value of σ for which σT is a non-singular transformation of one of the groups G_r, G_r' ...., associated with Γ_r, the transformation $\mathbf{T}$ is a singular transformation of Γ_r, and, consequently, this group contains singular transformations.

---

* Lie, "Continuerliche Gruppen," p. 501.

Let equations (A) (page 70), define the transformations of the subgroup G_r of the general linear homogeneous group. Then equations (B) will define the transformations of the subgroup Γ_r of the general projective group. If, now, G_r contains a singular transformation T, in order to determine whether Γ contains singular transformations, we may ascertain directly whether the transformation $\mathbf{T}$, corresponding to T, is a singular transformation of Γ_r. This method often seems preferable, and, in general at least, presents no difficulty.

Let G_r denote a subgroup of the general linear homogeneous group in n variables. The infinitesimal transformations $X_1, \ldots . X_j, \ldots . X_r$ of G_r satisfy relations of the form*

$$(X_i, X_k) \equiv \sum_1^r c_{iks} X_s, \qquad (i, k = 1, 2, \ldots . , r)$$

with constant coefficients c_{iks}. Now, it often happens that the coefficients of one (or more) of the X's, say X_j, is zero in each of these parenthesis-expressions, i. e.,

$$c_{ikj} = 0. \qquad (i, k = 1, 2, \ldots . , r)$$

In other words, X_j may not occur in any of the parenthesis-expressions for the X's taken in pairs.

Let us assume that $c_{ikj} = 0$ for $i, k = 1, 2, \ldots . , r$. Then

$$Y_1, Y_2, \ldots . , Y_r,$$

where

$$Y_k \equiv X_k, \qquad (k \neq j)$$
$$Y_j \equiv X_j + U$$

$\left(U\right.$ denoting the perspective transformation $\left. x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots . + x_n \frac{\partial}{\partial x_n}\right)$ are infinitesimal transformations of a subgroup G'_r of the general linear homogeneous group in n variables;* and, if

$$(Y_i, Y_k) \equiv \sum_1^r c'_{iks} Y_s,$$

we have $c'_{iks} = c_{iks}$. Therefore, if T_a, T_b, T_c are three transformations of G_r generated respectively by

$$a_1 X_1 + a_2 X_2 + \ldots . + a_r X_r,$$
$$b_1 X_1 + b_2 X_2 + \ldots . + b_r X_r,$$
$$c_1 X_1 + c_2 X_2 + \ldots . + c_r X_r,$$

---

* Lie, "Continuerliche Gruppen," p. 391.

and such that $T_b T_a = T_c$, the three transformations T'_a, T'_b, T'_c of G'_r generated respectively by

$$\begin{gathered} a_1 Y_1 + a_2 Y_2 + \dots + a_r Y_r, \\ b_1 Y_1 + b_2 Y_2 + \dots + b_r Y_r, \\ c_1 Y_1 + c_2 Y_2 + \dots + c_r Y_r, \end{gathered}$$

will satisfy the relation $T'_b T'_a = T'_c$, and conversely. Now, the six transformations T_a, T'_a, etc., belong to the general linear homogeneous group. Moreover, U is an infinitesimal transformation of the general linear homogeneous group. Therefore, by theorem I, T'_a may be considered as the resultant of the composition of T_a and a transformation generated by $a_j U$. In the notation adopted, the latter is represented by e^{a_j}, since it is defined by the equations

$$x'_i = x_i + a_j U x_i + \dots = e^{a_j} x_i. \qquad (i = 1, 2, \dots, n)$$

Consequently, $$T'_a = e^{a_j} T_a.$$

Similarly, $$T'_b = e^{b_j} T_b,$$ $$T'_c = e^{c_j} T_c.$$

Therefore, $$e^{c_j} T_c = T'_c = T'_b T'_a = e^{a_j} e^{b_j} T_b T_a = e^{a_j + b_j} T_c.$$

Whence it follows that

$$e^{c_j} = e^{a_j + b_j};$$

and so we may put $c_j = a_j + b_j$.

We thus obtain the following theorem, which, on account of the close relation existing between the general linear homogeneous group in n variables and the general projective group in $n - 1$ variables, applies also to the general projective group in $(n - 1)$ variables:

THEOREM IV.—*If T_a, T_b, and T_c, generated respectively by $\sum_1^r a_s X_s$, $\sum_1^r b_s X_s$, and $\sum_1^r c_s X_s$, and connected by the relation $T_b T_a = T_c$, are three transformations of the group $X_1, \dots X_j, \dots X_r$; and if, moreover, in the parenthesis-expressions*

$$(X_i, X_k) \equiv \sum_1^r c_{iks} X_s, \qquad (i, k = 1, 2, \dots, r)$$

the coefficients of X_j *are zero, that is, if*

$$c_{ikj}=0\,, \qquad (i,\,k=1,\,2,\,\ldots,\,r)$$

we may then put $c_j=a_j+b_j$.

In the case of the linear homogeneous groups, the process of finding the equations of the finite transformations of a given group, $X_1,\,\ldots,\,X_r$, can often be very greatly simplified. For let T be a transformation of G_r generated by the infinitesimal transformation $\sum_1^r l_s X_s$. To the symbol $\sum_1^r l_s X_s$ add and subtract the same multiple of U, where U denotes the perspective transformation,

$$x_1\frac{\partial}{\partial x_1}+\ldots\ldots+x_n\frac{\partial}{\partial x_n}.$$

Then, after rearranging and simplifying the terms, let

$$\sum_1^r l_s X_s\equiv\overline{X}+\varepsilon U,$$

where $\overline{X}$ is the symbol of an infinitesimal transformation, namely, $\sum_1^r l_s X_s-\varepsilon U$. Whether $\overline{X}$ is a transformation of the group G_r is immaterial; it is sufficient that $\overline{X}$ is an infinitesimal transformation of the general linear homogeneous group, of which G_r is a subgroup. Since

$$(\overline{X},\,\varepsilon U)\equiv 0,$$

we may, by theorem I, consider T as the resultant of the two commutative transformations generated respectively by $\overline{X}$ and εU. According to our notation, the transformation generated by εU may be represented by e^ε, since it is defined by the system of equations

$$x_i'=e^\varepsilon x_i. \qquad (i=1,\,2,\,\ldots\ldots,\,n)$$

Consequently, if S be the transformation generated by $\overline{X}$, the transformation T is identical with the transformation $e^\varepsilon S$, or

$$T=e^\varepsilon S.$$

In other words, if we know the system of equations defining S, we multiply the right-hand members of these equations by the quantity e^ε in order to obtain the system of equations that define the transformation T. By a proper choice of ε, $\overline{X}$ often takes such a form that it is much simpler first to find the transformation S than to obtain the transformation T directly. For example, suppose we wish to find the finite transformations generated by the infinitesimal transformations of the two-parameter group, xq, $xp+\alpha yq$. Here

$$\sum_1^r l_s X_s \equiv l_1 xq + l_2 (xp + \alpha yq) \equiv l_1 xq + l_2 (\alpha - 1) yq + l_2 U;$$

and we have $\overline{X} \equiv l_1 xq + l_2 (\alpha - 1) yq$, and $\varepsilon U \equiv l_2 U$. The transformation S generated by $\overline{X}$ is defined by the infinite series

$$\begin{aligned} x' &= x + [l_1 xq + l_2 (\alpha - 1) yq]\, x + \ldots\ldots; \\ y' &= y + [l_1 xq + l_2 (\alpha - 1) yq]\, y + \ldots\ldots; \end{aligned}$$

and can readily be summed. The transformation S multiplied by e^{l_2} gives us the required transformation T. The transformation T, considered as generated by $l_1 xq + l_2 (xp + \alpha yq)$, is defined by the infinite series

$$\begin{aligned} x_1 &= x + [l_1 xq + l_2 (xp + \alpha yq)]\, x + \ldots\ldots, \\ y_1 &= y + [l_1 xq + l_2 (xp + \alpha yq)]\, y + \ldots\ldots \end{aligned}$$

These series are, however, more difficult to sum than the series defining S.

At the end of this paper, a list is given of those two- and three-parameter subgroups of the general projective group in two variables, and of the general linear homogeneous group in three variables, that contain singular transformations, and are, therefore, not properly continuous. They are included among the subgroups of the general projective group and of the general linear homogeneous group enumerated by Lie on pages 288 and 519 respectively of his "Continuerliche Gruppen." The following groups have been selected from this list.

Example 1. The subgroup, xq, $xp + \alpha yq$ $(\alpha \neq 0, 1)$. For this group $\sum_1^r l_s X_s \equiv \overline{X} + \varepsilon U \equiv l_1 xq + l_2 (\alpha - 1) yq + l_2 U$, and equations (3) becomes

$$\begin{aligned} x' &= e^{l_2} x, \\ y' &= \frac{l_1 e^{l_2}}{l_2(\alpha - 1)} \left(e^{l_2(\alpha - 1)} - 1\right) x + e^{l_2 \alpha} y. \end{aligned}$$

Consequently, T_a and T_b are defined respectively by the equations

$$\left.\begin{aligned} x' &= e^{a_2} x, \\ y' &= \frac{a_1 e^{a_2}}{a_2(\alpha-1)}\left(e^{a_2(\alpha-1)}-1\right)x + e^{a_2\alpha} y, \end{aligned}\right\} \qquad \text{(A)}$$

and

$$\left.\begin{aligned} x'' &= e^{b_2} x', \\ y'' &= \frac{b_1 e^{b_2}}{b_2(\alpha-1)}\left(e^{b_2(\alpha-1)}-1\right)x' + e^{b_2\alpha} y'. \end{aligned}\right\} \qquad \text{(B)}$$

Eliminating x' and y' between equations (A) and (B), we obtain, as the system of equations defining $T_b\,T_a$,

$$\left.\begin{aligned} x'' &= e^{a_2+b_2} x, \\ y'' &= Nx + e^{(a_2+b_2)\alpha} y, \end{aligned}\right\} \qquad \text{(C)}$$

where N denotes

$$\frac{1}{\alpha-1}\left\{\frac{b_1 e^{b_2}}{b_2}\left(e^{b_2(\alpha-1)}-1\right)e^{a_2} + \frac{a_1 e^{a_2}}{a_2}\left(e^{a_2(\alpha-1)}-1\right)e^{b_2\alpha}\right\}.$$

Moreover, T_c is defined by the system of equations

$$\left.\begin{aligned} x'' &= e^{c_2} x, \\ y'' &= \frac{c_1 e^{c_2}}{c_2(\alpha-1)}\left(e^{c_2(\alpha-1)}-1\right)x + e^{c_2\alpha} y. \end{aligned}\right\} \qquad \text{(D)}$$

Wherefore, if $T_b\,T_a = T_c$,

$$\begin{aligned} e^{c_2} &= e^{a_2+b_2}, \\ e^{c_2\alpha} &= e^{(a_2+b_2)\alpha}, \\ \frac{c_1 e^{c_2}}{c_2(\alpha-1)}&\left(e^{c_2(\alpha-1)}-1\right) = N. \end{aligned}$$

From the first two of these equations, we obtain

$$\begin{aligned} c_2 &= a_2 + b_2 + 2n\pi i, \\ c_2\alpha &= (a_2+b_2)\alpha + 2n'\pi i, \end{aligned}$$

where n and n' are integers. Whence we have as a condition to which the integers n and n' are subject,

$$n\alpha = n'.$$

If α is irrational, this relation is possible only if both n and n' are zero. In this case equations (2) become

$$\left.\begin{aligned} c_2 &= a_2 + b_2, \\ c_1 &= \frac{(a_2 + b_2)(\alpha - 1) N}{e^{a_2+b_2}\left(e^{(a_2+b_2)(\alpha-1)} - 1\right)}. \end{aligned}\right\} \qquad \text{(E)}$$

Whence it follows that c_2 is finite for every set of finite values of the a's and b's; but if $a_2 + b_2 = \dfrac{2m\pi i}{\alpha - 1}$ (m being an integer $\neq 0$), c_1 will be infinite, provided the a's and b's are so chosen that $N \neq 0$.

Therefore, if α is irrational, the transformation $T_b\, T_a$ is singular, provided the a's and b's are so chosen that $N \neq 0$ and that $a_2 + b_2$ shall be an even multiple of πi other than zero. Substituting $\dfrac{2m\pi i}{\alpha - 1}$ in (C) for $a_2 + b_2$, we have, as the equations of definition of the singular transformation $T_b\, T_a$,

$$\begin{aligned} x'' &= e^{\frac{2m\pi i}{\alpha-1}} x, \\ y'' &= Nx + e^{\frac{2m\alpha\pi i}{\alpha-1}} y. \end{aligned} \qquad (N \neq 0,\ m \text{ an integer} \neq 0)$$

Let α be rational. In this case n is subject only to the condition that $n\alpha$ shall be an integer; and, if we put $\alpha = \dfrac{\mu}{\nu}$, where μ and ν are two integers relatively prime, $n = \lambda\nu$, where λ is an arbitrary integer. We now have

$$\begin{aligned} c_2(\alpha - 1) &= (a_2 + b_2 + 2n\pi i)(\alpha - 1) \\ &= (a_2 + b_2)\frac{\mu - \nu}{\nu} + 2\lambda(\mu - \nu)\pi i. \end{aligned}$$

And, therefore, equations (2) become

$$\begin{aligned} c_2 &= a_2 + b_2 + 2\lambda\nu\pi i, \\ c_1 &= \frac{\left[(a_2 + b_2)\dfrac{\mu - \nu}{\nu} + 2\lambda(\mu - \nu)\pi i\right] N}{e^{a_2+b_2}\left(e^{(a_2+b_2)\frac{\mu-\nu}{\nu}} - 1\right)}. \end{aligned}$$

As before, c_2 is finite for finite values of the a's and b's. But, if $a_2 + b_2 = \dfrac{2m\nu\pi i}{\mu - \nu}$ (m an integer), c_1 is infinite, provided $N \neq 0$, unless we can so choose λ that

$$\lambda(\mu - \nu) + m = 0;$$

in which case $c_1 = N$.

Consequently, if α is rational and equal to $\frac{\mu}{\nu}$, where μ and ν are two integers relatively prime, the transformation $T_b\,T_a$ defined by the equations

$$x' = e^{\frac{2m\nu\pi i}{\mu-\nu}}\,x,$$
$$y' = Nx + e^{\frac{2m\mu\pi i}{\mu-\nu}}\,y, \qquad (N \neq 0)$$

is singular, unless m contains $\mu-\nu$, in which case $T_b\,T_a$ is non-singular.

Let α be rational and equal to $\frac{\mu}{\nu}$; and let the a's and b's be so chosen that $T_b\,T_a$ is singular. Apply the transformation $T_b\,T_a$ successively $|\mu-\nu|$ times to the manifold (x, y). The resulting transformation is defined by the equations

$$x' = x,$$
$$y' = N'x + y,$$

where, denoting $e^{\frac{2m\nu\pi i}{\mu-\nu}}$ and $e^{\frac{2m\mu\pi i}{\mu-\nu}}$ respectively by ρ and σ, $N' = N(\rho^{\mu-\nu-1} + \rho^{\mu-\nu-2}\sigma + \dots + \rho\sigma^{\mu-\nu-2} + \sigma^{\mu-\nu-1})$. This transformation is, however, non-singular, and can be generated by the infinitesimal transformation $N'q + 0\,(xp + ayq)$, as may be seen on substituting in equation (D) $c_1 = N'$ and $c_2 = 0$. Thus, although $T_b\,T_a$ may be singular, if α is rational, some power of $T_b\,T_a$ is non-singular.* The significance of this fact will appear later.

Example 2. $\qquad xq,\ xp + q.$

The transformation, T_c, is defined by the equations

$$\left.\begin{aligned} x' &= e^{c_2}x, \\ y' &= \frac{c_1}{c_2}(e^{c_2} - 1)\,x + y + c_2. \end{aligned}\right\} \qquad \text{(A)}$$

And, if $T_b\,T_a = T_c$, we have also

$$\left.\begin{aligned} x' &= e^{a_2+b_2}x, \\ y' &= Nx + y + (a_2 + b_2), \end{aligned}\right\} \qquad \text{(B)}$$

where N is a function of the a's and b's and may be taken arbitrarily.

---

* Cf. Study, Leipzige Berichte, 1892. Taber, Proc. Lond. Math. Soc., vol. XXVI; Math. Ann., vol. XLVI; Bul. N. Y. Math. Soc., July, 1894.

Equations (B) may be obtained either in a manner similar to that in which equations (C) in example 1 were obtained, or they may be obtained immediately by means of theorem IV. We have $(xq, xp+q) = -xq$; and, consequently, if $T_b T_a = T_c$, we may put $c_2 = a_2 + b_2$. We may assume, therefore, that $T_b T_a$ is defined by equations (B) where N is a function of the a's and b's, and may be taken arbitrarily. In most cases I found the latter method preferable.

Comparing the two sets of equations (A) and (B), we obtain

$$c_2 = a_2 + b_2,$$
$$c_1 = \frac{N(a_2+b_2)}{e^{a_2+b_2}-1}.$$

From these it follows that every branch of c_1 is infinite for $a_2 + b_2 = 2m\pi i$ (m being an integer $\neq 0$), provided the a's and b's are so chosen that $N \neq 0$. Consequently, the given group contains singular transformations.

Example 3. $x_3p_2,\ x_1p_2,\ x_3p_1 + x_2p_2 + \alpha U.$

Here $\sum_1^r l_s X_s \equiv \bar{X} + \epsilon U \equiv l_1 x_3 p_2 + l_2 x_1 p_2 + l_3(x_3 p_1 + x_2 p_2) + l_3 \alpha U$, and the transformation T_c is defined by the equations

$$x_1' = e^{c_2\alpha} x_1 + c_3 e^{c_2\alpha} x_3,$$
$$x_2' = \frac{c_2 e^{c_2\alpha}}{c_3}(e^{c_2}-1)x_1 + e^{c_2(\alpha+1)}x_2 + e^{c_2\alpha}\left(\frac{c_1+c_3}{c_3}(e^{c_2}-1) - c_2\right)x_3,$$
$$x_3' = e^{c_2\alpha} x_3.$$

If $T_c = T_b T_a$, T_c is defined also by the equations

$$x_1' = e^{(a_2+b_2)\alpha} x_1 + (a_3+b_3) e^{(a_2+b_2)\alpha} x_3,$$
$$x_2' = N x_1 + e^{(a_2+b_2)(\alpha+1)} x_2 + M x_3,$$
$$x_3' = e^{(a_2+b_2)\alpha} x_3,$$

where M and N are functions of the a's and b's and may be taken arbitrarily. Wherefore,

$$e^{c_2\alpha} = e^{(a_2+b_2)\alpha},$$
$$c_3 = a_3 + b_3,$$
$$e^{c_2\alpha}\frac{c_2}{c_3}(e^{c_2}-1) = N,$$
$$e^{c_2\alpha}\frac{c_1}{c_3}(e^{c_2}-1) + \frac{e^{c_2\alpha}c_3}{c_3}(e^{c_2}-1) - c_2 e^{c_2\alpha} = M.$$

Whence,

$$c_3 = a_3 + b_3,$$

$$c_2 = \frac{N(a_3+b_3)}{e^{(a_3+b_3)\alpha}(e^{a_3+b_3}-1)},$$

$$c_1 = \frac{M - N + c_2\, e^{(a_3+b_3)\alpha}}{e^{(a_3+b_3)\alpha}(e^{a_3+b_3}-1)}(a_3+b_3).$$

From these equations it follows that every branch of both c_2 and c_1 is infinite if $a_3 + b_3 = 2m\pi i$ (m being an integer $\neq 0$), provided $N \neq 0$. Moreover, c_1 is then infinite of an order higher than c_2; and if $N = 0$, c_1 is infinite, provided $M + c_2\, e^{2m\alpha\pi i} \neq 0$.

Consequently, the given group contains singular transformations. The group associated with this group, namely, $q, xq, p + yq$, likewise contains singular transformations.

Example 4.—The subgroup, $x_3 p_1$, $x_3 p_2$, $\alpha x_1 p_1 + \beta x_2 p_2 + \vartheta x_3 p_3$. The transformation T_c is defined by the equations

$$x_1' = e^{c_3 \alpha} x_1 + \frac{c_1\, e^{c_3 \vartheta}}{c_3(\alpha - \vartheta)}(e^{c_3(\alpha-\vartheta)} - 1)\, x_3,$$

$$x_2' = e^{c_3 \beta} x_2 + \frac{c_2\, e^{c_3 \vartheta}}{c_3(\beta - \vartheta)}(e^{c_3(\beta-\vartheta)} - 1)\, x_3,$$

$$x_3' = e^{c_3 \vartheta} x_3.$$

If $T_c = T_b\, T_a$, T_c is also defined by

$$x_1' = e^{(a_3+b_3)\alpha} x_1 + M x_3,$$

$$x_2' = e^{(a_3+b_3)\beta} x_2 + N x_3,$$

$$x_3' = e^{(a_3+b_3)\vartheta} x_3.$$

Wherefore,

$$e^{c_3 \alpha} = e^{(a_3+b_3)\alpha},$$

$$e^{c_3 \beta} = e^{(a_3+b_3)\beta},$$

$$e^{c_3 \vartheta} = e^{(a_3+b_3)\vartheta},$$

$$\frac{c_1\, e^{c_3 \vartheta}}{c_3(\alpha - \vartheta)}(e^{c_3(\alpha-\vartheta)} - 1) = M,$$

$$\frac{c_2\, e^{c_3 \vartheta}}{c_3(\beta - \vartheta)}(e^{c_3(\beta-\vartheta)} - 1) = N.$$

From the first three of these equations, we obtain

$$c_3 \alpha = (a_3 + b_3)\alpha + 2n\pi i,$$

$$c_3 \beta = (a_3 + b_3)\beta + 2n'\pi i,$$

$$c_3 \vartheta = (a_3 + b_3)\vartheta + 2n''\pi i,$$

where n, n' and n'' are integers. Since these equations are simultaneous, we have the relations

$$\frac{n}{\alpha} = \frac{n'}{\beta} = \frac{n''}{\vartheta} \equiv \rho.$$

For certain values of α, β and ϑ (e. g., if α is rational, β and ϑ irrational), these relations are possible only if n, n' and n'', and therefore ρ are zero. With the foregoing meaning assigned to ρ, equations (2) become

$$c_3 = a_3 + b_3 + 2\rho\pi i,$$
$$c_2 = \frac{(\beta - \vartheta)\, N(a_3 + b_3 + 2\rho\pi i)}{(e^{(a_3+b_3)\vartheta}\,(e^{(a_3+b_3)(\beta-\vartheta)} - 1)},$$
$$c_1 = \frac{(\alpha - \vartheta)\, M(a_3 + b_3 + 2\rho\pi i)}{e^{(a_3+b_3)\vartheta}\,(e^{(a_3+b_3)(\alpha-\vartheta)} - 1)}.$$

From these equations it follows that every branch of c_2 is infinite if $a_3 + b_3 = \frac{2m\pi i}{\beta - \vartheta}$ (provided $N \neq 0$), unless $\rho + \frac{m}{\beta - \vartheta} = 0$, in which case c_2 is finite. But if $\rho = -\frac{m}{\beta - \vartheta}$, $\frac{m\alpha}{\beta - \vartheta}$, $\frac{m\beta}{\beta - \vartheta}$, and $\frac{m\vartheta}{\beta - \vartheta}$ are all integers. Consequently, if $a_3 + b_3 = \frac{2m\pi i}{\beta - \vartheta}$, and if, moreover, $\frac{m\alpha}{\beta - \vartheta}$, $\frac{m\beta}{\beta - \vartheta}$ and $\frac{m\vartheta}{\beta - \vartheta}$ are not all integers, every branch of c_2 is infinite (provided $N \neq 0$), and the corresponding transformation is singular. Similarly, every branch of c_1 is infinite if $a_3 + b_3 = \frac{2m\pi i}{\alpha - \vartheta}$, and if, moreover, $\frac{m\alpha}{\alpha - \vartheta}$, $\frac{m\beta}{\alpha - \vartheta}$, and $\frac{m\vartheta}{\alpha - \vartheta}$, are not all integers and $M \neq 0$.

Example 5.—The special linear homogeneous group

$$x_1 p_2, \; x_1 p_1 - x_2 p_2, \; x_2 p_1,$$

The transformation T_c is defined by the equations

$$x_1' = \left[\frac{e^d + e^{-d}}{2} + \frac{c_2}{d}\left(\frac{e^d - e^{-d}}{2}\right)\right] x_1 + \frac{c_3}{d}\left(\frac{e^d - e^{-d}}{2}\right) x_2,$$
$$x_2' = \frac{c_1}{d}\left(\frac{e^d - e^{-d}}{2}\right) x_1 + \left[\frac{e^d + e^{-d}}{2} - \frac{c_2}{d}\left(\frac{e^d - e^{-d}}{2}\right)\right] x_2,$$
$$x_3' = x_3,$$

where $d = \sqrt{c_2^2 + c_1 c_3}$.

Let $T_b\,T_a$ be defined by the equations

$$x_1' = a_{11}x_1 + a_{12}x_2, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1,$$
$$x_2' = a_{21}x_1 + a_{22}x_2,$$
$$x_3' = x_3.$$

Then, if $T_b\,T_a = T_c$,

$$e^d + e^{-d} = a_{11} + a_{22}, \tag{A}$$

$$c_1\left(\frac{e^d - e^{-d}}{d}\right) = 2a_{21}, \tag{B}$$

$$c_2\left(\frac{e^d - e^{-d}}{d}\right) = a_{11} - a_{22}, \tag{C}$$

$$c_3\left(\frac{e^d - e^{-d}}{d}\right) = 2a_{12}, \tag{D}$$

where

$$a_{11}a_{22} - a_{12}a_{21} = 1 \tag{E}$$

and

$$d = \sqrt{c_2^2 + c_1c_3}. \tag{F}$$

By means of equations (B), (C) and (D), we obtain the equation

$$(c_2^2 + c_1c_3)\left(\frac{e^d - e^{-d}}{d}\right)^2 = (a_{11} - a_{22})^2 + 4a_{21}a_{12},$$

whence

$$(e^d - e^{-d})^2 = (a_{11} + a_{22})^2 - 4, \qquad \text{(by (E) and (F))}$$

or

$$e^d - e^{-d} = \pm\sqrt{(a_{11} + a_{22})^2 - 4}. \tag{G}$$

Adding (A) and (G),

$$e^d = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4}}{2}.$$

Taking the logarithms of both sides,

$$d = \log\frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4}}{2} + 2k\pi i, \tag{H}$$

where k is an integer.

Since a_{11} and a_{22} are functions of the a's and b's, d is a multivalued function of the a's and b's.

Moreover, from equations (B), (C) and (D) we obtain

$$c_1 = \frac{2a_{21}}{e^d - e^{-d}}\, d,$$

$$c_2 = \frac{a_{11} - a_{22}}{e^d - e^{-d}}\, d,$$

$$c_3 = \frac{2a_{12}}{e^d - e^{-d}}\, d,$$

where d is defined by (H).

From (H), it follows that $d = 2k\pi i$, if $a_{11} + a_{22} = 2$. But if $d = 2k\pi i$, c_1 will be infinite (provided $a_{21} \neq 0$), unless we put $k = 0$, in which case $c_1 = a_{21}$, and, then, at least one branch of c_1 is finite. Similarly, at least one branch of c_2 and c_3 respectively is finite. Consequently, if $a_{11} + a_{22} = 2$, the transformation $T_b\, T_a$ is non-singular.

On the other hand, if $a_{11} + a_{22} + 2 = 0$, $d = (2k + 1)\pi i$. But in this case, c_1 will be infinite for any integer k, provided $a_{21} \neq 0$; and consequently, if $a_{21} \neq 0$, every branch of c_1 will be infinite. Similarly, every branch of c_2 will be infinite, provided $a_{11} \neq a_{22}$; and every branch of c_3 will be infinite, provided $a_{12} \neq 0$.

Let $a_{11} + a_{22} = -2$. Squaring,

$$a_{11}^2 + a_{22}^2 + 2a_{11}\, a_{22} = 4.$$

From (E),

$$4a_{11}\, a_{22} = 4 + 4a_{12}\, a_{21}.$$

Subtracting,

$$(a_{11} - a_{22})^2 = -4a_{12}\, a_{21}.$$

If, therefore, a_{12} or a_{21} is zero, $a_{11} = a_{22}$, and c_2 will not be infinite. Consequently, *if* $a_{11} + a_{22} + 2 = 0$, *and if, moreover,* a_{12} *and* a_{21} *are not both zero, the transformation* $T_b\, T_a$ *is singular.*

We know that a point p of general position in the space $(x_1 \ldots . x_n)$ will describe a continuous curve under the successive application of the infinitesimal transformation that generates a given transformation T. This continuous curve is called the *path-curve of the point* p *with respect to, or associated with, the infinitesimal transformation* in question, and is invariant with respect to this infinitesimal transformation, and, therefore, invariant with respect to the transformation T. It is obvious that there are ∞^{n-1} path-curves associated with any infinitesimal transformation that generates T (or ∞^{n-2}, if the space $(x_1 \ldots . x_n)$ is $(n-1)$-way extended).[*]

Let $X_1, X_2, \ldots . , X_r$ be the infinitesimal transformations of the group G_r. Let T be a transformation of this group, and

$$\sum_1^r c_j X_j = \sum_1^n {}_s\, \xi_s (x_1 \ldots . x_n,\ c_1 \ldots . c_r) \frac{\partial}{\partial x_s}$$

be the infinitesimal transformation that generates T. The path-curve of the point p with respect to this infinitesimal transformation may be termed the *path-curve of the point p associated with the transformation T*. The invariant curves, or path-curves, associated with T are given, therefore, by the system of simultaneous differential equations,[†]

$$\frac{dx_1}{\xi_1(x,c)} = \frac{dx_2}{\xi_2(x,c)} = \ldots . = \frac{dx_n}{\xi_n(x,c)}. \tag{A}$$

Let the $(n-1)$ independent integrals be

$$\left.\begin{array}{l} \phi_1 \ (x_1 \ldots . x_n,\ c_1 \ldots . c_r) = K_1, \\ \phi_2 \ (x_1 \ldots . x_n,\ c_1 \ldots . c_r) = K_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_{n-1} (x_1 \ldots . x_n,\ c_1 \ldots . c_r) = K_{n-1}. \end{array}\right\} \tag{B}$$

The equations (B), by giving all possible values to the constants of integration, $K_1 \ldots . K_{n-1}$, define the ∞^{n-1} of invariant curves, or path-curves associated with T. But when T is singular, no infinitesimal transformation of the group to which T belongs will generate T, and, therefore, there is no path-curve associated with T.[‡] Nevertheless, there are always, even in this case, curves invariant to T.

If we let $c_1 \ldots . c_r$ take all possible sets of values, equations (B) define the path-curves or invariant curves associated with the transformations of the given group G_r. In the case of a singular transformation, one (or more) of the parameters, say c_j, is infinite. Suppose $\phi_i = K_i$ contains the parameter c_j, then if c_j is put infinite in this equation, certain of the terms in ϕ_i may be infinite.

---

[*] Lie, "Differential Gleichungen," p. 300.

[†] Lie, Ibid., p. 301.

[‡] That is, no one of the path-curves generated by infinitesimal transformations of G_r is associated with T.

If the highest order of infinity of the terms thus rendered infinite is q, we may choose K_i infinite also of the q^{th} order. If, then, we divide through by a quantity infinite of the q^{th} order, the terms finite, or infinite, of an order lower than the q^{th}, will disappear. Proceeding similarly with the other equations, we obtain equations defining the curves invariant to those singular transformations of the given group for which $c_j = \infty$. If the equations (B) are all algebraic, that is, if all the path-curves or invariant curves associated with the infinitesimal transformations of G_r are algebraic, each of the curves invariant to a singular transformation T, in such cases as I have examined, break up into a finite system of lines (real or imaginary); and the application of the singular transformation T to a point p on one of these lines (not itself invariant) will convey p from the line in question to another of the same system. Moreover, the transformation repeated a certain number of times will convey the given point p back to the original line. Here we have a geometrical reason why some power of a singular transformation may often be non-singular; namely, some power of a singular transformation T applied to a point p on one of the lines of the system invariant, as a whole, to T, may move the given point along that line, which transformation may be effected by the repetition of an infinitesimal transformation of G_r.

On the other hand, if the path-curves, or invariant curves (or at least some of them), are transcendental, it often happens that each of the curves invariant to a singular transformation consists of an infinite number of lines; and the application of the singular transformation to a point p on one of these lines (not itself invariant) will convey p from that line to another line of the system; but, repeated any number of times, it will not, as in the preceding case, convey p back to the original line. In some cases each curve invariant to a singular transformation may even reduce to a single line.

I shall investigate the invariant curves of the first three groups considered above.

Example 1.—The invariant curves (path-curves) associated with the transformations of the group xq, $xp + ayq$ are given by the differential equation

$$\frac{dx}{c_2 x} = \frac{dy}{c_1 x + c_2 a y}. \tag{A}$$

Integrating, and denoting the constant of integration by K, we obtain the equation

$$x^{a} = \frac{c_1}{K}x + \frac{c_2}{K}(a-1)y, \tag{B}$$

which, by giving proper values to c_1 and c_2, defines the curves invariant to the transformation T of the given group. If, now, T is singular, then c_1 is infinite (see p. 76 *et seq.*); but since K is an arbitrary constant, we may choose K so that $\frac{c_1}{K} = C$, where C is finite, in which case $\frac{c_2}{K}$ is zero. The equation (B) thus becomes

$$x^{a} = Cx, \tag{C}$$

which represents a system of lines invariant as a whole to T.* The application of the singular transformation T to a point p of general portion on one of these lines (not itself invariant to T) will convey p from that line to another line of the system. For example (see p. 78), let $m = 1$ and $a = 3$. Then the singular transformation T is defined by

$$x' = -x,$$
$$y' = Nx - y, \qquad (N \neq 0)$$

and the equation of the invariant curves reduces to

$$x^{3} = Cx,$$

which breaks up into

$$x = +C, \quad x = -C,$$

and the special invariant $x = 0$.

Moreover, the application of T to a point p of general position on the line $x = +C$ will convey that point to the line $x = -C$. Applied again, it will reconvey the point p to the line $x = +C$. The second power of T, therefore, moves the point p along the line $x = +C$; and this transformation may be effected continuously by the repetition of an infinitesimal transformation of our group; that is to say, T^2 is non-singular.

---

* Each of the lines of this system is separately invariant to an infinitesimal transformation of G_r; namely, that for which $c_1 = 1$, $c_2 = 0$. For the path-curves of this infinitesimal transformation are defined by the equation $x =$ constant.

Example 2.—The invariant curves of the group xq, $xp + q$ are given by the differential equation

$$\frac{dx}{c_2 x} = \frac{dy}{c_1 x + c_2}. \tag{A}$$

Integrating, and denoting the constant of integration by K, we obtain

$$c_2 y = c_1 x + c_2 \log x + K. \tag{B}$$

For a singular transformation, c_1 is infinite (see p. 79). Choosing K properly, equation (B) becomes

$$x = C. \tag{C}$$

Each curve invariant to the transformation T reduces, therefore, to a single line, if T is singular.

Example 3.—The invariant curves of the group $x_3 p_2$, $x_1 p_2$, $x_3 p_1 + x_2 p_2$ are given by the simultaneous differential equations

$$\frac{dx_1}{c_3 x_3} = \frac{dx_2}{c_1 x_3 + c_2 x_1 + c_3 x_2} = \frac{dx_3}{0}. \tag{A}$$

Integrating, and denoting the constants of integration by K and C, we obtain the equations

$$\begin{aligned} e^{-\frac{x_1}{x_3}} x_2 &= -\frac{c_2}{c_3} x_3 e^{-\frac{x_1}{x_3}} \left(\frac{x_1}{x_3} + 1\right) - \frac{c_1}{c_3} x_3 e^{-\frac{x_1}{x_3}} + K, \\ x_3 &= C. \end{aligned} \tag{B}$$

For a singular transformation, c_1 is infinite of an order higher than c_2. Consequently, equations (B) reduce to

$$e^{\frac{x_1}{x_3}} = K' x_3,$$
$$x_3 = C,$$

from which we obtain

$$\frac{x_1}{x_3} = C' + 2k\pi i,$$
$$x_3 = C;$$

or

$$x_1 = x_3 C', \quad x_3 = C,$$
$$x_1 = x_3 (C' + 2\pi i), \quad x_3 = C,$$

. . . .

$$x_1' = x_3 (C' + 2m\pi i), \quad x_3 = C,$$

. . . .

Consequently, if T is singular, each curve invariant to T consists of an infinite number of lines. Moreover, the singular transformation applied to a point p on one of these lines (not itself invariant) will convey p from that line to another of the system.

The singular transformation is defined by the equations

$$\begin{aligned} x_1' &= x_1 + 2m\pi i x_3, \\ x_2' &= N x_1 + x_2 + M x_3, \qquad (N \neq 0) \\ x_3' &= x_3. \end{aligned}$$

This transformation applied to the point (CC', x_2, C) on the line

$$x_1 = x_3 C', \quad x_3 = C$$

will convey that point to the point $(CC' + 2m\pi i C, x_2', C)$ which, evidently, is a point on the line

$$x_1 = x_3 (C' + 2m\pi i), \quad x_3 = C.$$

Two points, p and p_1, of general position on the same smallest invariant manifold relative to the group G_r can always be interchanged by one or more transformations of G_r. In general, each of the transformations by which p and p_1 can be interchanged, can be generated by an infinitesimal transformation of G_r; in which case I shall say that the points p and p_1 can be continuously interchanged by the transformations of this group. But if G_r contains singular transformations, it sometimes happens that the points p and p_1 cannot be interchanged by a transformation of G_r that can be generated by an infinitesimal transformation of G_r; and, in this case, I shall say that the points p and p_1 cannot be continuously interchanged.

If the smallest invariant manifold relative to the r-parameter group G_r is q-way extended, $q \leqq r$, then there are ∞^{r-q} transformations of G_r that will interchange two points, p and p_1 of general position on any invariant manifold relative to G_r. If $r = q$, then there is at least *one* transformation, and in the case of certain groups there may be ∞^1.[*] If there is but one, and if this transformation is singular, that is, if this transformation cannot be generated by an infinitesimal transformation of G_r, then, clearly, not all points on each smallest invariant manifold can be continuously interchanged. But if there are more transformations than one that will interchange two points p and p_1 of general position, then it is by no means certain, when G_r contains singular transformations, that p and p_1 can be chosen so that all these transformations are singular; or, if $q < r$, that the ∞^{r-q} transformations are singular. In fact, in all cases I have considered, this is never possible. It may happen that but one, or all but one, of these transformations are singular. In this case the points of general position on any smallest invariant manifold can be continuously interchanged by means of the transformation of the given group, although the group contain transformations that cannot be generated by an infinitesimal transformation of the group.

---

[*] Lie, "Continuerliche Gruppen," p. 482, Lie here states that there is but one transformation if the group is simply transitive. But this is not always true, as is shown by the 2nd example below.

Most of the conclusions and examples which follow have been given by the author in a "Note on the Projective Group." Proc. Amer. Acad., vol. XXXIII, p. 498.

Associated with a group G_r with r-parameters is a subgroup $\mathbf{G}_\rho$ with $\rho \leqq r$ parameters of the general linear homogeneous group in r-variables—the adjoined of the group G_r. If these r-variables be considered as Cartesian coordinates, then every point in the space, S_r, to which the transformations of $\mathbf{G}_r$ are applied, represents a transformation of the group G_r. On the other hand, if the r-variables be considered as homogeneous coordinates, then every point in the space S_{r-1}, to which the transformations of $\mathbf{G}_\rho$ are applied, represents a one-parameter subgroup of G_r. Moreover, two points p and p_1 of general position on the same smallest invariant manifold in S_r (or S_{r-1}) relative to $\mathbf{G}_\rho$, represent transformations (or one-parameter subgroups) of G_r of the same type. For these points can be interchanged by means of one or more transformations of $\mathbf{G}_\rho$; and, therefore, the transformations (or one-parameter subgroups) represented by these points, may be transformed into each other by means of the transformations of G_r. If, however, the points p and p_1 cannot be continuously interchanged, the transformations (or one-parameter subgroups) represented by these points, although of the same type, are differently related from two transformations (or one-parameter subgroup) represented by points in S_r (or S_{r-1}) that can be continuously interchanged.*

I have examined all the two- and three-parameter groups enumerated by Lie in the "Continuerliche Gruppen," pp. 288 and 519. In each case the adjoined group $\mathbf{G}_\rho$ is such that two points of general position on the same smallest invariant manifold relative to $\mathbf{G}_\rho$ can always be interchanged continuously, notwithstanding that in certain cases the associated group $\mathbf{G}_\rho$ contains singular transformations. I have, therefore, as yet, found no group G_r whose transformations (or one-parameter subgroups) of the same type cannot be continuously transformed into each other by means of transformations of G_r; and it seems by no means certain that such groups exist.

---

* Proc. Am. Acad., vol. XXXIII, p. 498.

Let S and T be two transformations of the group G_r, and let S be transformed into S' by means of the transformation T, that is, let

$$TST^{-1} = S'.$$

If T is non-singular, S may be continuously transformed into S' by means of the infinitesimal transformation of G_r that generates T. But if T is singular, there is no infinitesimal transformation of G_r that generates T. Now, in such group as I have examined, T may always be considered as the resultant of two non-singular transformations of G_r. Let $T = T_b\, T_a$, in which case

$$TST^{-1} = T_b\, T_a\, S T_a^{-1}\, T_b^{-1} = S'.$$

If, now, T_a is commutative with S,

$$T_b\, T_a\, S T_a^{-1}\, T_b^{-1} = T_b\, S T_b^{-1} = S',$$

and, therefore, S may be continuously transformed into S' by means of the infinitesimal transformation of G_r that generates T_b. In every case examined, it is possible to consider T as the resultant of two non-singular transformations T_b and T_a, T_a being commutative with the given transformation S.

The following examples illustrate the effect of the existence of singular transformations among the transformations of a group G_r upon the interchange, by transformations of G_r, of points on the same invariant manifold relative to G_r. They have been selected from the list given at the end of this paper. The third group considered is the adjoined group of a number of three-parameter projective groups.

Example 1.—Consider the two-parameter projective group of the plane

$$xq,\ xp + 3yq.$$

The ∞^3 of non-singular transformations are defined by the system of equations

$$x' = e^{c_2} x_3,$$
$$y' = \frac{c_1}{2c_2}(e^{3c_2} - e^{c_1})x + e^{3c_2} y.$$

The group contains singular transformations which are defined by the system of equations

$$x' = -x,$$
$$y' = Nx - y. \qquad (N \neq 0)$$

Now, a singular transformation T applied to a point p on the line $x = +C$ will transform p to a point p_1 on the line $x = -C$; and, clearly, there is no non-singular transformation T_c among the transformations of the group that has the same effect. If the singular transformation T is applied to a point p on the special invariant $x = 0$, p will be conveyed across the invariant point $x = 0$, $y = 0$. But this can be done by a non-singular transformation whose path-curve is imaginary; for this transformation may be effected by the non-singular transformation

$$x' = -x,$$
$$y' = -y.$$

Therefore, two points p and p_1 in the plane, that lie on opposite sides of and equidistant from the special invariant $x = 0$, cannot *always* be interchanged among themselves continuously by means of the transformations of the group.

Example 2. xq, $xp + q$.

The ∞^3 of non-singular transformations are defined by the equations

$$x' = e^{c_2} x,$$
$$y' = \frac{c_1}{c_2}(e^{c_2} - 1)x + y + c_3.$$

The group contains singular transformations which are defined by the system of equations

$$x' = x,$$
$$y' = Nx + y + 2m\pi i. \qquad (N \neq 0,\ m \text{ an integer} \neq 0)$$

A ∞^1 of singular transformations applied to a given point p of general position on the line $x = C$ will convey that point to a given point p_1 of general position on the same line. Nevertheless, there is one non-singular transformation T_c that will do the same, namely,

$$x' = x,$$
$$y' = c_1 x + y.$$

Therefore, two points, p and p_1, of general position in the plane, may always be continuously interchanged by means of transformations of the given group.

Example 3. $x_3 p_2,\ x_1 p_2,\ x_1 p_1 + 2x_2 p_2.$

The ∞^3 of non-singular transformations are defined by the equations

$$\begin{aligned} x_1' &= e^{c_3} x_1, \\ x_2' &= \frac{c_2}{c_3}(e^{2c_3} - e^{c_3})\,x_1 + e^{2c_3} x_2 + \frac{c_1}{2c_3}(e^{2c_3} - 1)\,x_3, \\ x_3' &= x_3, \end{aligned}$$

and the ∞^2 of singular transformation are defined by the equations

$$\begin{aligned} x_1' &= -x_1, \\ x_2' &= Mx_1 + x_2 + Nx_3, \qquad (N \neq 0) \\ x_3' &= x_3. \end{aligned}$$

A ∞^1 of the singular transformations will convey a given point p of general position on the line $x_1 = +C$, $x_3 = k$, to a given point p_1 of general position on the line $x_1 = -C$, $x_3 = k$. Nevertheless, we can find one non-singular transformation that will do the same, namely,

$$\begin{aligned} x_1' &= -x_1, \\ x_2' &= Ax_1 + x_2, \\ x_3' &= x_3. \end{aligned}$$

For clearly, by a proper choice of A, this transformation T_c has the same effect when applied to a given point as the singular transformation T for any given values of M and N $(N \neq 0)$.

Example 4. $x_3 p_1,\ x_3 p_2,\ x_1 p_1 + 2x_2 p_2.$

The ∞^3 of non-singular transformations are defined by the equations

$$\begin{aligned} x_1' &= e^{c_3} x_1 + \frac{c_1}{c_3}(e^{c_3} - 1) x_3, \\ x_2' &= e^{3c_3} x_2 + \frac{c_2}{2c_3}(e^{3c_3} - 1) x_3, \\ x_3' &= x_3. \end{aligned}$$

The ∞^2 of singular transformations are defined by the equations

$$\begin{aligned} x_1' &= -x_1 + Mx_3, \\ x_2' &= x_2 + Nx_3, \\ x_3' &= x_3. \end{aligned} \qquad (N \neq 0)$$

By means of a singular transformation T a given point p of general position on the plane $x_3 = k$ can be transformed into a given point p_1 of general position in that plane. But it is easily seen that c_1, c_2, and c_3 can be chosen in ∞^1 of ways so that the transformation T_c will produce the same effect.

Example 5. $\quad x_3 p_1,\ x_3 p_2,\ 2x_1 p_2 + 3x_2 p_2 + x_3 p_3.$

The ∞^3 of non-singular transformations are defined by the equations

$$\begin{aligned} x_1' &= e^{2c_3} x_1 + \frac{c_1}{c_3}(e^{2c_3} - e^{c_3}) x_3, \\ x_2' &= e^{3c_3} x_2 + \frac{c_2}{2c_3}(e^{3c_3} - e^{c_3}) x_3, \\ x_3' &= e^{c_3} x_3. \end{aligned}$$

The ∞^2 of singular transformations are defined by the equations

$$\begin{aligned} x_1' &= x_1 + Mx_3, \\ x_2' &= -x_2 + Nx_3, \\ x_3' &= -x_3. \end{aligned} \qquad (N \neq 0)$$

The singular transformation T, if we regard x_1, x_2, x_3 as Cartesian coordinates, will convey a given point p of general position in the plane $x_3 = +C$ to a point p_1 on the plane $x_3 = -C$; and, clearly, there is no other transformation of the group that will do the same. The points on the special invariant $x_3' = 0$ can be continuously interchanged, for the transformation effected by T can also be effected by the non-singular transformation

$$\begin{aligned} x_1' &= x_1 + Ax_3, \\ x_2' &= -x_2, \\ x_3' &= -x_3. \end{aligned}$$

Therefore, in this group, points on opposite sides of, and equidistant from, the the special invariant plane $x_3 = 0$ cannot *all* be interchanged continuously among themselves.

The following groups enumerated by Lie on pp. 288 and 519 of his "Continuerliche Gruppen" are not properly continuous except in the neighborhood of the identical transformation.*

$$\boxed{q,\ p + xq,\ xp + 2yq} \qquad \boxed{q,\ xq,\ p + yq}$$

$$\boxed{p,\ q,\ xp + (y - x)q} \qquad \boxed{xq,\ xp - yq,\ yp}$$

$$\boxed{p,\ q,\ (a - 1)xp + ayq} \qquad \boxed{q,\ xq,\ xp + ayq}$$

$$\boxed{q,\ yq + p} \qquad \boxed{xq,\ xp + q}$$

$$\boxed{xq,\ xp + ayq\,(a \neq 0,\ 1)}$$

$$\boxed{x_3p_2,\ x_3p_1 + x_1p_2,\ x_2p_2 - x_3p_3 + \beta U} \qquad \boxed{x_3p_1,\ x_3p_2 + U,\ x_1p_1 + \beta U}$$

$$\boxed{x_3p_2,\ x_1p_2,\ x_3p_1 + x_2p_2 + \beta U} \qquad \boxed{x_3p_2,\ x_1p_3 + U,\ x_1p_1 + x_2p_2 + \beta U}$$

---

* Cf. Proc. Am. Acad., vol. XXXIII, p. 498.

$$\boxed{x_3p_1,\ x_3p_2,\ x_1p_1+x_1p_2+x_2p_2+\beta U}$$

$$\boxed{x_3p_2,\ x_3p_1+x_2p_2,\ U}$$

$$\boxed{x_1p_3,\ x_1p_1-x_3p_3,\ x_3p_1}$$

$$\boxed{x_1p_2;\ x_1p_1+x_3p_3,\ U}$$

$$\boxed{x_3p_1,\ x_3p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$

$$\boxed{x_3p_2,\ \alpha x_1p_1+\beta x_2p_2,\ U}$$

$$\boxed{x_3p_2,\ x_1p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$

$$\boxed{x_3p_2,\ x_3p_1+x_2p_2+\beta U}$$

$$\boxed{x_3p_2,\ x_1p_1+\alpha U,\ x_2p_2+\beta U}$$

$$\boxed{x_1p_2,\ x_1p_1+x_3p_3+\beta U}$$

$$\boxed{x_3p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$

Lines of Curvature on Annular Surfaces having Two Spherical Directrices.

BY VIRGIL SNYDER.

Let $x_1, x_2, \ldots, x_6$ be any six numbers which satisfy the quadratic identity

$$\Pi(x_1, x_2, \ldots, x_6) \equiv \Pi(x) \equiv 0;$$

these numbers will be taken as the homogeneous coordinates of a sphere. Suppose these coordinates to be functions of one parameter t; the spheres obtained by giving t all values in succession envelop an annular surface; the circle of intersection of two consecutive spheres lies on the surface and is a line of curvature. An annular surface may be defined as a surface having one system of lines of curvature composed of circles.

Consider the spheres which touch two consecutive spheres of the surface: these spheres constitute a special linear congruence; they must touch the sphere x in the points of its circle of intersection c with the consecutive sphere $x + dx$. The centres of the spheres of this congruence lie on a cone of revolution determined by c and the centre of x.

In case two consecutive spheres of the surface touch each other, their circle of intersection reduces to a point. The congruence is now degraded; it consists of the two hyperpencils of spheres containing one or the other of the two minimum lines determined by the point of tangency and the common tangent plane. On account of the close analogy between these points and pinch-points on ruled surfaces, I will apply the name pinch-points to define them.

The contact of two spheres x, y is expressed by the vanishing of the polar of x with regard to y, $\sum_{i=1}^{6} \frac{\partial \Pi}{\partial x_i} y_i = 0$. In particular, two consecutive spheres of the surface, x and $x + dx$ will be in contact if $\Pi(dx) = 0$. If the functions which define x are rational and of degree n, there are $2(n-2)$ pinch-points on the surface. If $\Pi(dx)$ vanishes identically, the surface reduces to a curve enveloped by spheres.

Now, suppose all of the spheres x to belong to a linear complex of spheres, $\psi = 0$. The spheres which touch x and belong to ψ must touch x in points of its trajectory circle with regard to ψ; this circle lies on the fundamental sphere s of ψ. Similarly, for the consecutive sphere $x + dx$ of the surface. The two corresponding trajectory circles intersect in two points P_1, P_2 which lie also on c. From these two points proceed two tangent pencils of spheres which belong to ψ and touch the spheres x, $x + dx$. The locus of each of these points is a line of curvature on the surface, for each locus lies on a sphere which everywhere cuts the surface at a constant angle; hence: *Every annular surface contained in a linear spherical complex has a line of curvature determined by the intersection of the fundamental sphere of the complex with the surface.* This curve can be found by rational operations. When a certain linear condition exists among the coefficients of ψ, this line of curvature lies in a plane.

Now, suppose that the spheres which envelop the surface belong to two linear complexes, ψ_1, ψ_2; then they belong to every complex of the system

$$\psi_1 + \lambda\psi_2 = 0,$$

and all the lines of curvature become rational, being the curve of intersection of the surface with the fundamental spheres $s_1 + \lambda s_2$. Among these complexes are two special ones, whose fundamental spheres s_1', s_2' are touched by all the spheres of the congruence. These two spheres touch the surface along lines of curvature. Their centres are nodal points on the surface of centres of the annular surface. The two directrices s_1', s_2' intersect in a circle $\varkappa$ which lies on every fundamental sphere $s_1 + \lambda s_2$. Among the fundamental spheres is one whose radius becomes infinite; it is the plane ϖ of the circle $\varkappa$.

All the lines of curvature of an annular surface containing two spherical directrices are determined by rational operations; every such surface must have one line of curvature of the second system which is a plane curve.

To see the relation of these lines of curvature, it is easiest (though not necessary) to interpret the properties of asymptotic lines of ruled surfaces having two rectilinear directrices by means of Lie's "Abbildung" [see vol. V of the Bulletin of the Amer. Math. Soc., p. 343]. The following theorems then result:

Each fundamental sphere $s_1 + \lambda s_2$ cuts every circle of curvature c in two points:

as λ varies, these points describe an involution on c, whose double points are the points of contact of c with s_1', s_2'.

All the lines of curvature of the second system touch each other at the pinch-points.

A simple geometrical construction shows that the locus of the centre of this involution is a curve lying wholly in the plane ϖ. Moreover, there are a number $(2n)$ of point spheres belonging to the annular surface; these points all lie on x; the locus in question passes through each of these points, and as no centre can lie within x, the locus touches x in $2n$ points. Its order is thus $2n$.

If two planes ϖ_1, ϖ_2 are contained in the pencil $s_1 + \lambda s_2$ of fundamental spheres, then all become planes, x becomes a straight line and all the lines of curvature are plane. The locus of the centre of the involution on c is the line x, axis of the pencil of planes.

If an annular surface has two plane directors, all of its lines of curvature are plane.

Surfaces of revolution are included in this category; the line x is now the axis of revolution, and the double points of the involution on c are imaginary.

The simplest type of this form of surface is that of Dupin's cyclides. The line x is the line joining the nodes.*

Now, suppose ψ_1 is the linear complex composed of the planes of space, its fundamental sphere is the plane at infinity. If ψ_2 is general, all the other fundamental spheres $s_1 + \kappa s_2$ are concentric with the directrix s_2'. As all of the generators of the annular surface are now planes, it becomes a developable; all the generating planes of this developable touch s_2'. The theorems regarding lines of curvature hold the same as before, hence:

If the planes which generate a developable surface all touch a fixed sphere, its (non-linear) lines of curvature are determined by its intersection with a family of spheres concentric with the fixed one.

An immediate deduction from this theorem is the following: If one line of curvature of a developable is a spherical curve, they all are, and the spheres are all concentric. In this form it was obtained by Paul Serret by means† of differential geometry.

---

* The second pair of nodes (always imaginary) is to be entirely ignored, as they belong to the second generation of the cyclide, and are not included in the envelope of spheres which define the surface.

† Théorie géométrique et méchanique des lignes de double courbure. Paris, 1860.

In particular, if the radius of s_2' becomes zero, all the planes pass through a point and the developable surface becomes a cone. The theorem then states the simple fact that the lines of curvature of a cone are determined by its lines of intersection with a family of spheres concentric with its vertex.

If ψ_1 is composed of the points of space, the annular surface becomes the curve wherein s_2 intersects the point locus of the third complex which defines the surface.

Finally, if ψ_1 is points, ψ_2 planes, the annular surface reduces to the line of curvature enveloped by minimum lines on the point locus.

When the two directrices coincide, a particularly interesting case is presented; now the involution becomes degraded as the double points coincide; one point of every pair of conjugates coincides with this double point, and every fundamental sphere contains the curve of contact with the directrix. It is, therefore, a circle, and evidently the circle $\varkappa$. The sphere s' is now itself usually a generator. The other fundamental spheres cut the surface in $\varkappa$ and in one other line of curvature. The first theorem for the general case has now lost its meaning. The locus of the centre of involution is now the circle $\varkappa$. The circle $\varkappa$ may become a straight line. The necessary modifications of the theorem for this case are obvious.

Analytically, one may proceed thus:

If the surface is defined by six equations of the form

$$x_i = \phi_i(t),$$

these values x_i are to be substituted in an equation of the form

$$\sum_{i=1}^{6} a_i x_i = 0,$$

and the a_i so determined that the equation

$$\sum a_i \phi_i = 0$$

is identically satisfied. When the degree of ϕ_i is 2, there are three independent complexes: the surface is a Dupin's cyclide. When ϕ_i is of degree 3, there are 2 independent complexes. The resulting surface is of order six (or five, and the plane at infinity). *Every annular surface of order* 6 *has only rational lines of curvature.* When ϕ_i is of degree 4, the surface belongs to a single linear complex (in general). The other lines of curvature can be determined by the method outlined by Picard.

If the surface is defined by complexes, there are four equations of condition

$$f(x) = 0, \quad \psi_1(x) = 0, \quad \psi_2(x) = 0, \quad \Pi(x) = 0$$

among the homogeneous parameters $x_1 \ldots . x_6$ of the equation

$$\lambda_0 (x^2 + y^2 + z^2) - 2\lambda_1 x - 2\lambda_2 y - 2\lambda_3 z + \lambda_4 = 0,$$

wherein the λ_i are all linear and homogeneous in $x_1 \ldots . x_6$, and the envelope of the sphere is required.

The determination of the point equation of the fundamental sphere of the complex

$$\psi_1 + \varkappa\psi_2 = 0$$

has been considered in a previous paper.*

These properties are sufficient to make a theorem which I proved some time ago more definite.† The theorem can now be completed by the statement:

In the latter case the surface is an annular surface whose generators are included among those that cut the fixed sphere at the constant angle.

CORNELL UNIVERSITY, *Sept.* 20, 1898.

---

* V. Snyder, "Determination of Nodes in Dupin's Cyclides." Annals of Mathematics, vol. XI, p. 137.

† V. Snyder, "Geometry of some Differential Expressions in Hexaspherical Coordinates." Bulletin Amer. Math. Society, vol. IV, p. 147, middle of page.

Remarks concerning the Expansions of the Hyperelliptic Sigma-Functions.

By Oskar Bolza.

The following remarks are supplementary to my two papers, "The Partial Differential Equations for the Hyperelliptic Θ- and σ-Functions," and "Proof of Brioschi's Recursion Formula for the Expansion of the Even σ-Functions of Two Variables," American Journal of Mathematics, vol. XXI, pp. 107–125, and pp. 175–190. I shall refer to them as B. I and B. II respectively, and shall use the same notation as there.*

If
$$R(x) = \phi(x)\,\psi(x) \tag{1}$$
be a decomposition of $R(x)$ into two factors of degree $\rho+1$, then†
$$F_0(x,\xi) = \tfrac{1}{2}\big(\phi(x)\,\psi(\xi) + \phi(\xi)\,\psi(x)\big) \tag{2}$$
is one of the possible forms of Weierstrass' function $F(x,\xi)$, since it satisfies the conditions (2) of B. I.

Hence equation (41) of B. I becomes
$$\sum_{\alpha,\,\beta}\left(\frac{\partial^2 \log \sigma_0}{\partial u_\alpha\,\partial u_\beta}\right)_0 g_\alpha(s)\,g_\beta(t) \equiv 0 \tag{3}$$

---

* The following misprints in these papers have come to my notice:
p. 109, line 14, read $x-a$ instead of $a-a$.
p. 115, line 12, read (17) instead of (27).
p. 118, line 4 from bottom, read $\overline{U}=U$, $\overline{\Gamma}=\Gamma$.
p. 122, foot-note, drop upper index (x).
p. 124, line 2, sign of first term on the right, + instead of —.
p. 124, equation (41), read $(t-s)^2$ instead of $(t-s)$.
p. 176, equation (5), read y^2 instead of y_2.
p. 176, equation (9), read $\lambda_{11}\,x\xi$ instead of $\lambda_{11}\,x$.
p. 176, foot-note, add: (H), (6), (7), (10).
p. 177, lines 5 and 6, the sign = ought to be on the right of the term following it.
p. 183, equation (18), read N_2^2 instead of N^2.

† See Baker, Amer. Journ. of Math., vol. XX, p. 337.

for all values of s and t, if we denote by $\mathfrak{S}_0$ the $\mathfrak{S}$-function of algebraic characteristic $\phi\psi$ associated with $F_0(x, \xi)$. Therefore, *the terms of dimension two will disappear from the expansion of $\mathfrak{S}_0$ into a power series.*

A special interest attaches, therefore, to the function $\mathfrak{S}_0$, and I propose to consider in the following note:

1). The partial differential equations for $\mathfrak{S}_0$.

2). The recursion formulæ for the expansion of $\mathfrak{S}_0$ for the lowest cases $\rho = 1$ and $\rho = 2$.

§1. *The Partial Differential Equation for $\mathfrak{S}_0$.*

Theorem I.—*The $\mathfrak{S}$-function of algebraic characteristic $\phi\psi$ associated with the function*

$$F_0(x, \xi) = \tfrac{1}{2}\left[\phi(x)\psi(\xi) + \phi(\xi)\psi(x)\right]$$

and denoted by $\mathfrak{S}_0$, satisfies the partial differential equation

$$\frac{\partial \mathfrak{S}_0}{\partial a} = -\tfrac{1}{2}\mathfrak{S}_0 \sum_{\alpha,\beta} \lambda_{\alpha\beta} u_\alpha u_\beta - \sum_{\alpha,\beta} \varkappa_{\alpha,\beta} u_\alpha \frac{\partial \mathfrak{S}_0}{\partial u_\beta} + \frac{1}{R'(a)} \sum_{\alpha,\beta} \frac{\partial^2 \mathfrak{S}_0}{\partial u_\alpha \partial u_\beta} g_\alpha(a) g_\beta(a); \tag{4}$$

the coefficients $\lambda_{\alpha\beta}$, $\varkappa_{\alpha\beta}$ being defined by the equations

$$R'(a)\Lambda(x, \xi) \equiv R'(a) \sum_{\alpha,\beta} \lambda_{\alpha\beta} g_\alpha(x) g_\beta(\xi) = \frac{\Phi(x, \xi)\Phi(a, a) - \Phi(x, a)\Phi(\xi, a)}{8(x-a)(\xi-a)}, \tag{5}$$

$$R'(a) \mathrm{K}(x, \xi) \equiv R'(a) \sum_{\alpha,\beta} \varkappa_{\alpha\beta} g_\alpha(x) h_\beta(\xi)$$
$$= \frac{(x-\xi)^{\rho-1}\Phi(a, a) - (a-\xi)^{\rho-1}\Phi(x, a)}{2(x-a)}, \tag{6}$$

$$(x-\xi)^{\rho-1} = \sum_\beta g_\beta(x) h_\beta(\xi), \tag{7}$$

where

$$\Phi(x, \xi) = \frac{\phi(x)\psi(\xi) - \phi(\xi)\psi(x)}{(x-\xi)}. \tag{8}$$

Proof: a). As a consequence of (3), equation (40) of B. I becomes

$$\frac{\partial \log \Theta(0, 0, \ldots\ldots, 0)}{\partial a} = -2\sum_{\alpha,\beta} \frac{g_\alpha(a) g_\beta(a) a_{\alpha\beta}}{R'(a)},$$

and, therefore, the second term on the right-hand side of equation (H) of B. I disappears.

b). The substitution of the special value (2) for $F(x, \xi)$ in B. I (7) and (10) furnishes

$$\Lambda(x, \xi) = \frac{\phi(x)\psi(\xi) - \phi(\xi)\psi(x)}{8(x-\xi)(x-a)(\xi-a)} - \frac{\phi(x)\phi(\xi)\psi^2(a)}{8R'(a)(x-a)^2(\xi-a)},$$

$$\mathrm{K}(x, \xi) = \tfrac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \tfrac{1}{2}\frac{(a-\xi)^{\rho-1}\phi(x)\psi(a)}{R'(a)(x-a)^2},$$

and if we notice that

$$\frac{\phi(x)\psi(a)}{x-a} = \Phi(x, a),$$

$$R'(a) = \Phi(a, a),$$

we obtain the above values (5) and (6).

From (4) follows immediately the

Corollary I:

If $\frac{T_n}{(2n)!}$ *denotes the term of order* $2n$ *in the expansion of* σ_0 *according to powers of* $u_1, u_2, \ldots, u_\rho$, *then*

$$\frac{\partial T_{n-1}}{\partial a} = -(n-1)(2n-3)\,T_{n-2}\cdot\sum_{\alpha,\beta}\lambda_{\alpha\beta}u_\alpha u_\beta$$
$$-\sum_{\alpha,\beta}\varkappa_{\alpha\beta}u_\alpha\frac{\partial T_{n-1}}{\partial u_\beta} + \frac{1}{2n(2n-1)R'(a)}\sum_{\alpha,\beta}\frac{\partial^2 T_n}{\partial u_\alpha \partial u_\beta}g_\alpha(a)g_\beta(a). \quad (9)$$

From the derivative $\frac{\partial T_{n-1}}{\partial a}$ we can pass to any Aronhold-process by the

Corollary II:

If

$$M(x) = \sum_{i=0}^{\rho+1}\binom{\rho+1}{i}M_i x^{\rho+1-i}$$

denotes an integral function of x *at most of degree* $\rho+1$, *and if*

$$\phi(x) = \sum_{i=0}^{\rho+1}\binom{\rho+1}{i}\phi_i x^{\rho+1-i},$$

then

$$\sum_{i=0}^{\rho+1}M_i\frac{\partial T_{n-1}}{\partial\phi_i} = (n-1)\frac{M_0}{\phi_0}T_{n-1} - \sum_{(a)}\frac{M(a)}{\phi'(a)}\frac{\partial T_{n-1}}{\partial a}, \quad (10)$$

the summation $\sum_{(a)}$ *extending over the* $\rho+1$ *roots of* $\phi(x)$.

Proof: $\sum_{i=0}^{\rho+1} M_i \frac{\partial T_{n-1}}{\partial \phi_i}$ can be obtained by replacing, in $M(z)$, $\binom{\rho+1}{i} z^{\rho+1-i}$ by $\frac{\partial T_{n-1}}{\partial \phi_i}$, for $i = 0, 1, 2, \ldots, \rho+1$. The same substitution changes

$$-\frac{\phi(z)}{z-a} \text{ into } \frac{\partial T_{n-1}}{\partial a} = \sum_{i=1}^{\rho+1} \frac{\partial T_{n-1}}{\partial \phi_i} \frac{\partial \phi_i}{\partial a},$$

since
$$\frac{\partial \phi(z)}{\partial a} = -\frac{\phi(z)}{z-a}, \text{ say } = \sum_{i=1}^{\rho+1} \binom{\rho+1}{i} B_i z^{\rho+1-i},$$

and, therefore,
$$\frac{\partial \phi_i}{\partial a} = B_i.$$

Now
$$M(z) - \frac{M_0}{\phi_0}\phi(z) = \sum_{(a)} \frac{M(a)}{\phi'(a)} \frac{\phi(z)}{z-a},$$

and, therefore, by the above substitution

$$\sum_{i=0}^{\rho+1} M_i \frac{\partial T_{n-1}}{\partial \phi_i} = \frac{M_0}{\phi_0} \sum_{i=0}^{\rho+1} \phi_i \frac{\partial T_{n-1}}{\partial \phi_i} - \sum_{(a)} \frac{M(a)}{\phi'(a)} \frac{\partial T_{n-1}}{\partial a}.$$

But T_{n-1} is a homogeneous function* of $\phi_0, \phi_1, \ldots, \phi_{\rho+1}$ of dimension $n-1$, provided the $g_\alpha(x)$'s are independent of these quantities; for the quantities

$$\omega_{\alpha\lambda}, \quad \eta_{\alpha\lambda}, \quad a_{\alpha\beta}, \quad b_{\alpha\beta}, \quad \tau_{\alpha\beta}$$

are homogeneous in the ϕ_i's and of dimensions

$$-\tfrac{1}{2}, \quad +\tfrac{1}{2}, \quad +1, \quad +\tfrac{1}{2}, \quad 0$$

respectively. Hence, it follows from the definition of σ_0 (cf. (18) and (39) of B. I) that

$$\sigma_0(u_\alpha;\, m\phi_i) = \sigma_0(m^{\frac{1}{2}} u_\alpha;\, \phi_i)$$

for every m, therefore,

$$\sum_{i=0}^{\rho+1} \phi_i \frac{\partial \sigma_0}{\partial \phi_i} = \tfrac{1}{2} \sum_{\alpha} u_\alpha \frac{\partial \sigma_0}{\partial u_\alpha},$$

and, consequently,
$$\sum_{i=0}^{\rho+1} \phi_i \frac{\partial T_{n-1}}{\partial \phi_i} = (n-1)\, T_{n-1},$$

which completes the proof of (10).

---

*Cf. Klein, "Hyperelliptische Sigmafunctionen II," Math. Ann., 32, p. 369. As I start from a different definition of the σ-functions, I wished to give an independent proof.

The connection between Klein's σ-function of characteristic $\phi\psi$ (denoted by σ) and the function σ_0 is stated in the following

Corollary III:

If we put $$F_0(x, \xi) - a_x^{\rho+1} a_\xi^{\rho+1}$$

(of which we know a priori that it is divisible by $(x-\xi)^2$*), equal to*

$$2(x-\xi)^2 \sum_{\alpha, \beta} d_{\alpha\beta}\, g_\alpha(x)\, g_\beta(\xi),$$

then σ *and* σ_0 *are connected by the relation*

$$\sigma_0 = e^{-\frac{1}{2}\sum\limits_{\alpha,\beta} d_{\alpha\beta} u_\alpha u_\beta} \sigma. \tag{11}$$

This follows immediately from the fact that σ is associated with

$$F(x, \xi) = a_x^{\rho+1} a_\xi^{\rho+1};$$

combined with equations (16), (25), (26) of my paper, "On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind," Papers read at the International Mathematical Congress, 1893.

From (11) follows that $\frac{1}{2}\sum\limits_{\alpha, \beta} d_{\alpha\beta}\, u_\alpha u_\beta$ is the term of the second order in the expansion of σ in confirmation of known results.*

§2.—*The Case* $\rho = 1$.

If we choose $g_1(x) = 1$ and put $T_n = c_n u^{2n}$, the differential equation (9) becomes

$$R'(a)\, \frac{\partial c_{n-1}}{\partial a} = \frac{1}{8}(n-1)(2n-3)\, R_{\phi\psi}\, c_{n-2} + (n-1)\, \phi_0\, \psi(a)\, c_{n-1} + c_n, \tag{12}$$

$R_{\phi\psi}$ denoting the resultant of ϕ and ψ. For the computation of λ_{11}, notice that $\Lambda(x, \xi)$ is of degree $\rho - 1 = 0$ in x and ξ; hence, it can be computed by giving x and ξ any particular values. Putting $x = \xi = a'$, the second root of ϕ, we obtain

$$\lambda_{11} = \frac{1}{8}\, \frac{\phi'(a')\, \psi(a')}{(a-a')^2} = -\frac{\phi_0^2\, \psi(a)\, \psi(a')}{R'(a)} = -\frac{1}{8}\, \frac{R_{\phi\psi}}{R'(a)}.$$

---

* Schröder, "Ueber den Zusammenhang der hyperelliptischen σ- und ϑ-Functionen, Göttinger, Diss., 1890 (74), (85).

From (12) follows:

$$\sum_{(a)} R'(a)\frac{\partial c_{n-1}}{\partial a} = \frac{(n-1)(2n-3)}{4} R_{\phi\psi}\, c_n + 2c_n + (n-1)\,\phi_0 \sum_{(a)} \psi(a).$$

But

$$\sum_{(a)} \phi_0 \psi(a) = \sum_{(a)} \left(\phi_0\psi(a) - \psi_0\phi(a)\right) = 4\left[\vartheta_1 - \frac{\phi_1}{\phi_0}\vartheta_0\right],$$

ϑ_0, ϑ_1, ϑ_2 being the coefficients of $\vartheta = (\phi, \psi)_1$.

On the other hand,

$$\sum_{(a)} R'(a)\frac{\partial c_{n-1}}{\partial a} = \sum_{(a)} \frac{\phi'^2(a)\,\psi(a)}{\phi'(a)}\frac{\partial c_{n-1}}{\partial a} = -2\Delta_\phi \sum_{(a)} \frac{\psi(a)}{\phi'(a)}\frac{\partial c_{n-1}}{\partial a},$$

Δ_ϕ being the discriminant of ϕ; but, according to (10),

$$-\sum_{(a)} \frac{\psi(a)}{\phi'(a)}\frac{\partial c_{n-1}}{\partial a} = \sum_{i=0}^{2} \psi_i \frac{\partial c_{n-1}}{\partial \phi_i} - (n-1)\frac{\psi_0}{\phi_0} c_{n-1},$$

hence,

$$\begin{aligned}\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial c_{n-1}}{\partial \phi_i} &- (n-1)\frac{\psi_0}{\phi_0}\Delta_\phi c_{n-1}\\ &= \frac{(n-1)(2n-3)}{8} R_{\phi\psi}\, c_{n-2} + c_n + 2(n-1)\left(\vartheta_1 - \frac{\phi_1}{\phi_0}\vartheta_0\right)c_{n-1}.\end{aligned}$$

A similar equation is obtained by interchanging ϕ and ψ. Add the two equations and notice that

$$\frac{\psi_0}{\phi_0}\Delta_\phi + \frac{\phi_0}{\psi_0}\Delta_\psi + 2\vartheta_0\left(\frac{\psi_1}{\psi_0} - \frac{\phi_1}{\phi_0}\right) = 2A_{\phi\phi},$$

where $A_{\phi\psi} = (\phi, \psi)_2$.

The result is the following

THEOREM II:

If $\sigma_\lambda(u)$ is that even σ-function which corresponds to the decomposition of the biquadratic $R(x)$ into the two quadratic factors $\phi\psi$, and if we write

$$\sigma_\lambda(u) = e^{-\frac{1}{2}e_\lambda u^2}\left[1 + c_2\frac{u^4}{4!} + c_3\frac{u^6}{6!} + \ldots\right], \tag{13}$$

then the coefficients c_2, c_3, ... are determined by the recursion-formula

$$c_n = \delta c_{n-1} - (n-1)\,A_{\phi\psi} c_{n-}{}^{1} - \frac{(n-1)(2n-3)}{8} R_{\phi\psi}\, c_{n-2}, \tag{14}$$

where δ denotes the following Aronhold-process:

$$\delta f = \tfrac{1}{2}\left[\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial f}{\partial \phi_i} + \Delta_\psi \sum_{i=0}^{2} \phi_i \frac{\partial f}{\partial \psi_i}\right] \tag{15}$$

and $$\Delta_\phi = (\phi, \phi)_2, \quad \Delta_\psi = (\psi, \psi)_2. \quad A_{\phi\psi} = (\phi, \psi)_2$$

and $R_{\phi\psi}$ denotes the resultant of ϕ and ψ.

Since $R_{\phi\psi}$ is a combinant of ϕ and ψ, we have at once

$$\delta R_{\phi\psi} = 0; \tag{16}$$

and the ordinary rules for Aronhold-process furnish easily

$$\delta A_{\phi\psi} = \Delta_\phi \Delta_\psi = A^2_{\phi\psi} - R_{\phi\psi}. \tag{17}$$

Hence, follows the

THEOREM III:

The coefficient c_n in the expansion of $\sigma_\lambda(u)$ are integral functions of the two invariants $A_{\phi\psi}$ and $R_{\phi\psi}$, and if we put

$$c_n = \sum_{i=0} \gamma^{(n)}_{n-2i} \quad R^i_{\phi\psi} \quad A^{n-2i}_{\phi\psi}, \tag{18}$$

with the agreement that $\gamma^{(n)}_{n-2i} = 0$ whenever $n - 2i < 0$, the coefficients $\gamma^{(n)}_{n-2i}$ are determined by the recursion-formulæ

$$\gamma^{(n)}_n = 0,$$

$$\gamma^{(n)}_{n-2i} = -2i\,\gamma^{(n-1)}_{n-1-2i} - (n+1-2i)\,\gamma^{(n-1)}_{n+1-2i} - \frac{(n-1)(2n-3)}{8}\,\gamma^{(n-2)}_{n-2i}. \tag{19}$$

The proof of (19) is immediate: substitute in (14) for c_n, c_{n-1}, c_{n-2} their expressions in terms of $A_{\phi\psi}$ and $R_{\phi\psi}$, apply (16) and (17), and equate corresponding terms on both sides.

To compare these results with those obtained by Weierstrass by an entirely different method (Werke, vol. II, p. 253), we have to take ϕ and ψ in the normal form:

$$\phi = 4z_2(z_1 - e_\lambda z_2),$$
$$\psi = z_1^3 + e_\lambda z_1 z_2 + e_\mu e_\nu z_2^2,$$

which furnishes

$$A_{\phi\psi} = -6e_\lambda, \quad R_{\phi\psi} = 4(12e_\lambda^2 - g_2). \tag{20}$$

Weierstrass arranges c_n according to powers of

$$12e_\lambda = -2A_{\phi\psi},$$

and $$2\varepsilon_\lambda = 6e_\lambda^2 - \tfrac{1}{2}g_2 = \tfrac{1}{8}R_{\phi\psi}.$$

Hence, in order to obtain accordance with Weierstrass' notation, we must replace

$$\gamma^{(n)}_{n-2i} \text{ by } \frac{(-2)^{n-2i}}{8^i}\, c_{n-2i,\, i},$$

after which substitution, formula (19) will exactly coincide with Weierstrass' recursion-formula (J).

§2.—*The Case* $\rho = 2$.

a). *Computation of* $\Lambda(x, \xi)$ *and* $\mathrm{K}(x, \xi)$.

For $\rho = 2$, $\Lambda(x, \xi)$ is the first polar of $\Lambda(x, x)$ with respect to ξ. According to (5),

$$R'(a)\,\Lambda(x, x) = \frac{\Phi(x, x)\,\Phi(a, a) - \Phi^2(x, a)}{8\,(x-a)^2};$$

but* $\Phi(x, y) = 3\vartheta_x^2\vartheta_y^2 + \tfrac{1}{2} J(xy)^2,$

where $\vartheta = (\phi, \psi)_1, \quad J = (\phi, \psi)_3$ (21)

and an easy symbolic computation, in which we have to make use of the relation*

$$J^2 = 6\,(\vartheta, \vartheta)_4,$$

leads to the result:

$$\Phi(x, x)\,\Phi(y, y) - \Phi^2(x, y) = 3\,[6H_x^2H_y^2 - J\vartheta_x^2\vartheta_y^2](xy)^2,$$

where $H = (\vartheta, \vartheta)_2.$ (21)

The function $6H - J\vartheta$ plays an important part in the theory of the cubic involution

$$\lambda\phi + \mu\psi;$$

it is that biquadratic to which all the cubics of the involution are apolar. For shortness, we denote it by G:

$$6H - J\vartheta = G. \tag{21}$$

Our result is then

$$R'(a)\,\Lambda(x, x) = \tfrac{3}{8}\, G_x^2\, G_a^2. \tag{22}$$

Hence, if we choose, as in B. II,

$$g_1(x) = \frac{x}{2}, \quad g_2(x) = \tfrac{1}{2}, \tag{23}$$

---

* A table of the principal formulæ concerning cubic involutions is given in §1 of my paper, "Die cubische Involution, etc." Math. Annalen, Bd. 50.

we have

$$R'(a)\sum_{\alpha,\beta}\lambda_{\alpha\beta}\,u_\alpha\,u_\beta = \tfrac{3}{8}\,G_u^2\,G_a^2. \tag{24}$$

Further, according to (6),

$$R'(a)\,\mathrm{K}(x,\xi) = \frac{(x-\xi)\,\Phi(a,a)-(a-\xi)\,\Phi(x,a)}{2(x-a)}$$

the right-hand side is an integral function of a of degree 3, which we denote by $P(a)$; the coefficient P_0 of a^3 is easily found to be

$$P_0 = -\tfrac{3}{8}\,\vartheta_0\,(x-\xi).$$

Hence we may write

$$R'(a)\,\mathrm{K}(x,\xi) = P(a) + \tfrac{3}{8}\,\frac{\vartheta_0}{\phi_0}\,(x-\xi)\,\phi(a),$$

and the right-hand side is now reduced to the second degree in a.

According to the definition (6) of the $\varkappa_{\alpha\beta}$'s, $\mathrm{K}(x,\xi)$ may be used as an abbreviation for

$$\sum_{\alpha,\beta}\varkappa_{\alpha\beta}\,u_\alpha\,\frac{\partial T_{n-1}}{\partial u_\beta},$$

inasmuch as it changes into this expression, when we replace $g_\alpha(x)$ by u_α, $h_\beta(\xi)$ by $\dfrac{\partial T_{n-1}}{\partial u_\beta}$. This substitution changes $(x-\xi)^{\rho-1}$ into

$$\sum_\beta u_\beta\,\frac{\partial T_{n-1}}{\partial u_\beta} = (2n-2)\,T_{n-1}.$$

With the same agreement applied to $P(a)$, we may write our differential equation (9) in the form

$$\begin{aligned}\frac{\partial T_{n-1}}{\partial a} = \frac{1}{R'(a)}\Bigg\{&-\frac{3(n-1)(2n-3)}{2}\,G_u^2\,G_a^2\,T_{n-2}\\ &+\frac{1}{8n(2n-1)}\sum_{\alpha,\beta}\frac{\partial^2 T_n}{\partial u_\alpha\,\partial u_\beta}\,a^{4-\alpha-\beta} - P(a) - \frac{3(n-1)\,\vartheta_0\,\phi(a)}{\phi_0}\,T_{n-1}\Bigg\},\end{aligned} \tag{25}$$

An analogous expression $(\overline{25})$ for the derivative of T_{n-1} with respect to a root b of ψ is derived from (25) by interchanging a and b, ϕ and ψ, which changes $G_u^2\,G_a^2$ into $G_u^2\,G_b^2$, $P(a)$ into $-P(b)$, ϑ_0 into $-\vartheta_0$.

b). *The recursion-formula for T_n.*

In order to derive from (25) a recursion-formula for T_n, we apply the same method as in B. II, §2: Multiply (25) by $\dfrac{\phi(t)\psi(a)}{t-a}$, t denoting an arbitrary parameter, and add with respect to the roots of ϕ; multiply $(\overline{25})$ by $\dfrac{\psi(t)\phi(b)}{t-b}$ and add with respect to the roots of ψ. Finally, add the two results thus obtained.

Since the right-hand side of (25) is an integral function of the second degree of a, we can apply the reasoning of §2 of B. II and obtain as the result of the process on the right:

$$-3(n-1)(2n-3)\,G_u^2\,G_t^2\,T_{n-2}+\frac{1}{4n(2n-1)}\sum_{\alpha,\beta}\frac{\partial^2 T_n}{\partial u_\alpha\,\partial u_\beta}\,t^{4-\alpha-\beta}$$
$$+\frac{3(n-1)\vartheta_0}{\phi_0\psi_0}\left(\phi_0\psi(t)-\psi_0\phi(t)\right)T_{n-1}.$$

The transformation of the left-hand side is given in §3 of B. II; but it can be very much simplified by the application of Corollary II of §1.

Since
$$\frac{\phi(t)\psi(a)}{t-a}=\Phi(t,a),$$
we may write
$$\sum_{(a)}\frac{\phi(t)\psi(a)}{t-a}\,\frac{\partial T_{n-1}}{\partial a}=\sum_{(a)}\frac{\phi'(a)\,\Phi(t,a)-\frac{4}{3}\phi(a)\,\Phi'(t,a)}{\phi'(a)}\,\frac{\partial T_{n-1}}{\partial a},$$
where $\Phi'(t,z)=\dfrac{\partial\Phi(t,z)}{\partial z}$ and the zero terms
$$-\tfrac{4}{3}\phi(a)\,\Phi'(t,a)$$
has been added to reduce the degree of the numerator to three.

Use symbolic notation and put
$$\Phi(t,z)=\tfrac{1}{3}\Phi_z^3, \tag{26}$$
then
$$\phi'(z)\,\Phi(t,z)-\tfrac{4}{3}\phi(z)\,\Phi'(t,z)=\tfrac{4}{3}\left[\phi_z^3\,\phi_1\,\Phi_z^3-\phi_z^3\,\Phi_z\,\Phi_1\right]=-\tfrac{4}{3}(\Phi\phi)\,\Phi_z\,\phi_z^2.$$

Hence, if we put, as in B. II,
$$3(\Phi\phi)\,\Phi_z\,\phi_z^2=M_z^3 \tag{27}$$

and apply (10), we obtain

$$\sum_{(a)} \frac{\phi(t)\psi(a)}{t-a} \frac{\partial T_{n-1}}{\partial a} = \tfrac{1}{2}\sum_{i=0}^{3} M_i \frac{\partial T_{n-1}}{\partial \phi_i} - \frac{(n-1)}{2} \frac{M_0}{\phi_0} T_{n-1}.$$

Similarly, if we put

$$-3(\Phi\psi)\Phi_z \psi_z^2 = N_z^3, \tag{27}$$

we have

$$\sum_{(b)} \frac{\psi(t)\phi(b)}{t-b} \frac{\partial T_{n-1}}{\partial b} = \tfrac{1}{2}\sum_{i=0}^{3} N_i \frac{\partial T_{n-1}}{\partial \psi_i} - \frac{(n-1)}{2} \frac{N_0}{\psi_0} T_{n-1}.$$

Hence, if we define the operator D by

$$D(f) = \sum_{i=0}^{3}\left(M_i \frac{\partial f}{\partial \phi_i} + N_i \frac{\partial f}{\partial \psi_i}\right), \tag{28}$$

and notice that

$$M_0\psi_0 + N_0\phi_0 = -3\Phi_0\vartheta_0,$$

we obtain on the left-hand side as final result of the above-described process:

$$\tfrac{1}{2} D(T_{n-1}) + \tfrac{3}{2}(n-1)\frac{\Phi_0\vartheta_0}{\phi_0\psi_0} T_{n-1}.$$

The second term cancels against the term

$$\frac{3(n-1)\vartheta_0}{\phi_0\psi_0}\left(\phi_0\psi(t) - \psi_0\phi(t)\right) T_{n-1}$$

on the right. For if in the equation

$$\phi(z)\psi(t) - \phi(t)\psi(z) = \tfrac{1}{2}(zt)\Phi_z^2,$$

understood homogeneously, we put

$$z_1 = 1, \quad z_2 = 0; \quad t_1 = t, \quad t_2 = 1,$$

we obtain

$$\phi_0\psi(t) - \psi_0\phi(t) = \tfrac{1}{2}\Phi_0.$$

Hençe, we have

$$D(T_{n-1}) = -6(n-1)(2n-3)\,G_u^2 G_t^2 T_{n-2} + \frac{1}{2n(2n-1)} \sum_{\alpha,\beta} \frac{\partial^2 T_n}{\partial u_\alpha \partial u_\beta} t^{4-\alpha-\beta}. \tag{29}$$

Replace t by $\frac{u_1}{u_2}$, multiply by u_2^2 and denote

$$u_2^2 (D(f))_{t=\frac{u_1}{u_2}} = D_0(f). \tag{30}$$

Make use of (11) and of the known value

$$\tfrac{2}{10}\,\Theta(u) = \tfrac{2}{10}\,(\phi\psi)^2\,\phi_u\,\psi_u \tag{31}$$

for the second term in the expansion of Klein's σ-function, then the following result* is obtained:

THEOREM IV.

The successive terms in the expansion of Klein's σ*-function* $\sigma_{\phi\psi}(u_1 u_2)$ *into the series*

$$\sigma_{\phi\psi}(u_1, u_2) = e^{\frac{2}{10}\Theta(u)}\left\{1 + \frac{T_2}{4!} + \frac{T_3}{6!} + \ldots\right\} \tag{31}$$

are obtained by the recursion-formula

$$T_n = D_0(T_{n-1}) + 6\,(n-1)(2n-3)\;G(u)\;T_{n-2}, \tag{32}$$

where the operator D_0, *the quadratic* $\Theta(u)$, *and the biquadratic* $G(u)$ *are defined by equations* (21), (26) *to* (28), (30) *and* (31).

UNIVERSITY OF CHICAGO, *December 2d*, 1899.

---

* Other recursion-formulæ for $\sigma_0(u_1, u_2)$ have been given by Brioschi, "Sulla teorica delle funzioni iperellittiche di primo ordine," Cap. VIII, Annali di Matematica (2), vol. 14, in particular p. 314; and Wiltheiss, "Differentialgleichungen und Reihenentwicklungen der Θ-functionen," §6, Math. Annalen, Bd. 29, in particular p. 298.

On a certain Class of Groups of Transformation in Space of Three Dimensions.

By H. F. Blichfeldt.

In his "Untersuchungen über die Grundlagen der Geometrie,"[*] Professor Lie has investigated a remarkable class of groups of transformations in three variables, defined by the following properties: two points have one, and only one, invariant; $s > 2$ points have no invariants independent of such two-point invariants.

This class of groups belongs to a wider class in n variables defined by the following properties: *Not less than* $m > 1$ *points may possess invariants, while* s *points,* $s > m$, *may have no invariants independent of the* m*-point invariants.*

This wider class will include the Group of Euclidean Motions in space of two or three dimensions, the Group of Translations in space of n dimensions, the Group of Euclidean Motions and Similar Transformations in space of three dimensions, etc.

There is a law for such groups which can be stated as follows:

If a finite group in n *variables has no invariants for less than* m *points* $x_1 y_1 \ldots .$, $x_2 y_2 \ldots .$, $\ldots .$, $x_m y_m \ldots .$, *and no invariants for* $s > m$ *points independent of the* m*-point invariants, then must the set of independent invariants in the* m *points* $x_1 y_1 \ldots .$, $x_2 y_2 \ldots .$, $\ldots .$, $x_m y_m \ldots .$ *contain all the coordinates of all these* m *points.*

For, if one of the coordinates, as x_1, were absent, it is obvious that all the corresponding coordinates $x_2, x_3, \ldots , x_m$ would be absent from the independent m-point invariants; and, since the invariants of s points, $s > m$, are all made up of the m-point invariants, the former would be free from the variables

---

[*] See Lie's "Theorie der Transformationsgruppen," vol. III, Abtheilung V, or "Leipziger Berichte," vol. XLII.

$x_1, x_2, \ldots, x_s$. Among the differential equations defining these invariants, we should then necessarily find the following:

$$\frac{\partial f}{\partial x_1}=0, \quad \frac{\partial f}{\partial x_2}=0, \ldots, \quad \frac{\partial f}{\partial x_s}=0; \tag{I}$$

s independent equations in all. But, the group considered being finite, containing ρ infinitesimal transformations, say, it could not give rise to more than ρ independent differential equations in any number of points. By choosing s sufficiently large, we see the impossibility of the system (I), and our assumption that one of the coordinates of the m points $x_1 y_1 \ldots, x_2 y_2 \ldots, \ldots, x_m y_m \ldots$ could be absent from the set of independent invariants in these m points is contradicted.

According to this law, it would, for example, be impossible so to choose the system of coordinates that the two-point invariant for the Euclidean Group of Motions in a plane, $(x_1-x_2)^2+(y_1-y_2)^2$, would contain less than the four coordinates $x_1 y_1$, $x_2 y_2$, which is otherwise apparent.

From this law can be made the following important deduction:

In all the groups that we are considering (defined above), no m infinitesimal transformations can have the same path-curves ("Bahncurven").[*]

In the following, Professor Lie's notation will be used. A group formed by ρ independent infinitesimal transformations, in three variables, say,

$$X_i f \equiv \xi_i(x, y, z)\frac{\partial f}{\partial x}+\eta_i(x, y, z)\frac{\partial f}{\partial y}+\zeta_i(x, y, z)\frac{\partial f}{\partial z}, \qquad i=1, 2, \ldots, \rho,$$

we shall call the "ρ-parametric group $X_1 f \ldots X_\rho f$." It is convenient to write "R_n" for "space of n dimensions."

Lie has determined all groups in R_3 of the class defined above for which $m=2$, as already mentioned. To these belong the Euclidean Group of Motions in R_3,

$$p, q, r, \quad yr-zq, \quad zp-xr, \quad xq-yp,$$

whose two-point invariant is the distance between the two points. (Here we have written p, q, r in place of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ respectively.)

---

[*] A particular case of this law was given by Lie. See pp. 404–405 of his "Theorie der Transformationsgruppen," vol. III.

If we examine the Euclidean Group of Motions and Similar Transformations in R_3,

$$p,\ q,\ r,\quad yr-zq,\quad zp-xr,\quad xq-yp,\quad xp+yq+zr,$$

we find it to be a group of the class considered above, for which $m=3$. Three points have two independent invariants, namely, two angles of the plane triangle with the three points as vertices. It might be of interest to find whether there are other groups of like character, i. e., *groups in three variables* x, y, z, *for which not less than three points have invariants, and for which the invariants*[*] *of* $s>3$ *points are all dependent upon the three-point invariants.*

Following the methods employed by Lie in his "Untersuchungen über die Grundlagen der Geometrie," and by paying attention to the laws given above for such groups, we find that they are *eight-*, *seven- or six-parametric* according as the number of independent three-point invariants is one, two or three.

The Eight-parametric Groups.

There are no eight-parametric primitive groups in three variables.[*] To such a group must, therefore, belong an invariant system of surfaces or curves.[†]

All groups in three variables, x, y, z, having a single invariant system of surfaces, but no invariant system of curves, and those having an invariant system of curves, say $x=$ const., $y=$ const., and no invariant system of surfaces of the form $\phi(x,y)=$ const., have been determined.[‡] Only the following two types are eight-parametric, and have no three infinitesimal transformations with the same path-curves:

$$\text{A}\ \boxed{\begin{array}{l} p,\ q,\ xq+r,\ xp-yq-2zr,\ yp-z^2r,\ xp+yq, \\ x^2p+xyq+(y-xz)r,\ xyp+y^2q+z(y-xz)r \end{array}}$$

$$\text{B}\ \boxed{\begin{array}{c} p,\ q,\ xq,\ xp-yq,\ yp,\ xp+yq+r, \\ x^2p+xyq+\tfrac{3}{2}xr,\ xyp+y^2q+\tfrac{3}{2}yr \end{array}}$$

---

[*] See chap. 7, vol. III, of "Theorie der Transformationsgruppen."

[†] We say that a group *has an invariant system of surfaces or curves*, or that such a system *belongs to the group*, when the members of that system are interchanged by the transformations of the group.

[‡] See chap. 8, vol. III, of "Theorie der Transformationsgruppen."

The remaining eight-parametric groups must have an invariant system of curves, say $x = \text{const.}$, $y = \text{const.}$, and an invariant system of surfaces $\phi(x, y) = \text{const.}$, —say $x = \text{const.}$ They must, therefore, be of the form

$$X_i f \equiv \xi_i(x)\, p + \eta_i(x, y)\, q + \zeta_i(x, y, z)\, r, \qquad i = 1, 2, \ldots, 8.$$

The shortened groups ("verkürzte Gruppe"),

$$\overline{X}_i f \equiv \xi_i(x)\, p + \eta_i(x, y)\, q, \qquad i = 1, 2, \ldots$$

can be written down immediately from the tables of such groups.* It may be noticed that it follows from the laws given above that these shortened groups must be six-, seven- or eight-parametric, and that the shortened groups

$$\overline{\overline{X}}_i f \equiv \xi_i(x) p, \qquad i = 1, \ldots$$

must be three-parametric.

The remaining terms $\zeta_i(x, y, z)\, r$ of the infinitesimal transformations of these groups are then determined as Professor Lie has determined the corresponding terms in the groups considered in his "Untersuchungen über die Grundlagen der Geometrie." We would find the following groups:

$$\Gamma \quad \boxed{\begin{array}{c} p, q, \ xq + r, \ x^2 q + 2xr, \ x^3 q + 3x^2 r, \ x^4 q + 4x^3 r, \\ xp + 2yq + zr, \ x^2 p + 4xyq + (2xz + 4y)\, r \end{array}}$$

$$\Delta \quad \boxed{\begin{array}{c} p, q, \ xq + r, \ x^2 q + 2xr, \ x^3 q + 3x^2 r, \ yq + zr, \\ xp + yq, \ x^2 p + 3xyq + (xz + 3y)\, r \end{array}}$$

$$\mathrm{E} \quad \boxed{\begin{array}{c} r, p, q, \ xq, \ x^2 q + xr, \ yq + zr, \ xp + yq, \\ x^2 p + 2xyq + yr \end{array}}$$

$$\mathrm{Z} \quad \boxed{\begin{array}{c} r, p, q, \ xq, \ xr, \ 2xp + yq + zr, \ yq + (ay + bz)\, r, \\ x^2 p + xyq + xzr; \ b \neq 0, \ a(b - 1) = 0 \end{array}}$$

---

* See page 861 of Lie's "Continuerliche Gruppen."

The criterion that these groups possess the required properties resolves itself into the following conditions:

If

$$X_i^{(n)}f \equiv \xi_i(x_n, y_n, z_n)p_n + \eta_i(x_n, y_n, z_n)q_n + \zeta_i(x_n, y_n, z_n)r_n, \quad i = 1, 2, \ldots, 8$$

be any one of the groups written in the variables x_n, y_n, z_n, the eight partial differential equations

$$X_i^{(1)}f + X_i^{(2)}f + X_i^{(3)}f = 0, \qquad i = 1, 2, \ldots, 8$$

should be independent of each other.

If $I(a, b, c)$ represents the invariant in the points x_a, y_a, z_a; x_b, y_b, z_b; x_c, y_c, z_c, the four invariants

$$I(1, 2, 3), \quad I(1, 2, 4), \quad I(1, 3, 4), \quad I(2, 3, 4)$$

should be independent of each other.

The groups Α and Γ, and Ζ, when $b = -1$, do not satisfy the last condition.*

Thus, *the eight-parametric groups satisfying the required conditions stated above are the groups* Β, Δ *and* Ε, *and the group*

$$\mathrm{Z}' \quad \boxed{\begin{array}{c} r, p, q, \ xq, \ xr, \ 2xp + yq + zr, \ yq + (ay + bz)r, \\ x^2p + xyq + xzr; \ b \neq 0 \text{ or } -1, \ a(b-1) = 0. \end{array}}$$

Here x, y, z may be regarded as complex variables. If we seek all typical groups with *real variables*, we would still have to find the real groups similar to the groups just given. The principles involved in the solution of this problem are clearly exhibited in Chapter 19 of Lie's "Theorie der Transformationsgruppen," vol. III. The real group-types are found to be

---

* It is interesting to notice that in these cases the invariants $I(a, b, c)$ may be put in such form that we have identically

$$I(1, 2, 3) = I(1, 2, 4) + I(2, 3, 4) + I(3, 1, 4).$$

the groups B, Δ and E as given, Z′ with the additional condition that a and b are real constants, and the group

$$\text{H}\quad \boxed{\begin{array}{c} r,\ p,\ q,\ xq,\ xr,\ 2xp + yq + zr,\ x^2p + xyq + xzr, \\ yq + zr + c(zq - yr)\ ;\ c \text{ real}, \end{array}}$$

which is similar to the group Z′ by means of an imaginary transformation of the variables.

The Seven-parametric Groups.

The seven-parametric groups are more numerous. Here three points have two independent invariants, which, together, must contain all the nine coordinates of the three points ; and no three infinitesimal transformations of any one of the groups can have the same path-curves. By attending to these laws and proceeding as in the case of the eight-parametric groups, we find the seven-parametric groups without much difficulty.

The only primitive group is the before-mentioned Group of Euclidean Motions and Similar Transformations

$$\text{A}\quad \boxed{p,\ q,\ r,\ yr - zq,\ zp - xr,\ xq - yp,\ xp + yq + zr}$$

From Chapter 8 of "Theorie der Transformationsgruppen," vol. III, we find the following groups:

$$\text{B}\quad \boxed{p,\ r,\ q + xr,\ xq + \tfrac{1}{2}x^2r,\ xp - yp,\ yp + \tfrac{1}{2}y^2r,\ xp + yq + 2zr}$$

$$\Gamma\quad \boxed{p,\ q,\ r,\ xq,\ xp - yq,\ yp,\ zr + a(xp + yq)}$$

The remaining groups, about 25 in all, can be put in the form

$$X_i f \equiv \xi_i(x)p + \eta_i(x, y)q + \zeta_i(x, y, z)r, \qquad i = 1, 2, \ldots, 7,$$

where the shortened groups

$$\overline{X}_i f \equiv \xi_i(x)p + \eta_i(x, y)q, \qquad i = 1, 2, \ldots$$

are five-, six- or seven-parametric, and the shortened groups

$$\overline{\overline{X}}_i f \equiv \xi_i(x)\, p, \qquad i = 1, \ldots .$$

are two- or three-parametric.

The writer has not determined all the real groups belonging to this class, but has determined the *real primitive groups*, i. e., groups with no real invariant systems of surfaces or curves.* There are only two types of such groups, namely, the group A and one similar to this:

$$\mathrm{A}' \quad \boxed{p,\ q,\ r,\ yr + zq,\ xr + zp,\ xq - yp,\ xp + yq + zr}$$

The Six-parametric Groups.

The six-parametric groups under consideration are very numerous, being all the groups in three variables for which two points have no invariants and for which three points have three.† There are no primitive groups of this class, and, therefore, no real primitive.*

General Properties of these Groups.

There being no invariant relation connecting the coordinates of two points, any point in general position is free to move in any way in space if one point is fixed. All of the groups, excepting two, are imprimitive, and we should, therefore, expect that certain systems of points are restricted to move on certain loci if a given point is fixed.

In the case of the eight-parametric groups, if two points in general position are fixed, a third point, also in general position, must move on the surface

$$I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x, y, z) = I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3),$$

where x_1, y_1, z_1; x_2, y_2, z_2 are the two fixed points, x_3, y_3, z_3 the third point; x, y, z running coordinates, and $I(x_1$, etc.) the invariant in the three points.

---

*The writer has found that in R_3 all real primitive groups must be similar to those that are fully primitive, i. e., possess no invariant systems of curves or surfaces, real or imaginary.

†In fact, a ρ-parametric group in R_n will always belong to the wider class of groups defined above if ρ is a multiple of n, and there are no invariants for less than $\rho/n + 1$ points.

In the case of the seven-parametric groups, if two points in general position are fixed, any third point in general position must move on the curve

$$\begin{aligned} &I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x, y, z) = I(x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3), \\ &J(x_1, y_1, z_1;\ \text{etc.} \qquad\qquad\quad) = J(x_1, y_1, z_1;\ \text{etc.} \qquad\qquad\quad), \end{aligned}$$

where I and J are the two three-point invariants, x_1, y_1, z_1, etc., as above. Thus, if two points are fixed in the case of the Euclidean Group of Motions and Similar Transformations, any third point must move on a circle whose plane is perpendicular to the straight line joining the two fixed points, and whose center lies in that line. This is also geometrically evident.

In the case of the six-parametric groups, if two points in general position are fixed, all other points are fixed.

STANFORD UNIVERSITY, CAL.

Canonical Form of a Linear Homogeneous Substitution in a Galois Field.

By Leonard Eugene Dickson.

---

1. A simple canonical form of the general m-ary linear homogeneous substitution with integral coefficients taken modulo p, a prime, has been obtained by M. Jordan by a rather lengthy analysis.* The method may be readily generalized to apply to substitutions in an arbitrary Galois field. The present paper, however, gives a short proof by induction of the generalized theorem. That the new method is of practical value in actually reducing a given substitution to its canonical form is illustrated by the example of §3.

In §§4–7, the explicit form of all substitutions in the $GF[p^n]$ commutative with a given substitution in that field is set up. In particular, the number of such substitutions is deduced, the result being in accord with that of M. Jordan† for the case $n = 1$.

In §8 is given a simple criterion for the conjugacy of two linear homogeneous substitutions in a Galois field.

2. Consider an m-ary substitution with coefficients in the $GF[p^n]$,

$$S:\quad \xi_i' = \sum_{j=1}^{m} \alpha_{ij}\,\xi_j. \qquad (i = 1, \ldots, m)$$

In order that S shall multiply by a constant K the linear function

$$\eta \equiv \sum_{i=1}^{m} \lambda_i \xi_i,$$

---

*"Traité des substitutions," pp. 114–126.

† Ibid., p. 136.

the following conditions must be satisfied:

$$\sum_{i=1}^{m} \lambda_i \alpha_{ij} = K\lambda_j. \qquad (j = 1, \ldots, m) \quad (1)$$

Hence K must be a root of the characteristic equation

$$\Delta(K) \equiv \begin{vmatrix} \alpha_{11} - K & \alpha_{12} & \ldots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} - K & \ldots & \alpha_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mm} - K \end{vmatrix} = 0.$$

Corresponding to each root K, we can determine at least one set of solutions λ_i of the equations (1), and hence, at least one function η.

If $\Delta(K) = 0$ has m distinct roots $K_1, K_2, \ldots, K_m$ (in general not belonging to the initial Galois field), we reach m linear functions $\eta_1, \eta_2, \ldots, \eta_m$, which S multiplies by $K_1, K_2, \ldots, K_m$ respectively. These functions are linearly independent. For, if constants μ_i exist such that

$$\mu_1\eta_1 + \mu_2\eta_2 + \ldots + \mu_m\eta_m \equiv 0,$$

we have, on applying the substitutions $1, S, S^2, \ldots, S^{m-1}$, the identities

$$K_1^r\mu_1\eta_1 + K_2^r\mu_2\eta_2 + \ldots + K_m^r\mu_m\eta_m \equiv 0.$$
$$(r = 0, 1, \ldots, m-1)$$

The determinant $|K_i^r|$ equals the product of the differences of $K_1, \ldots, K_m$, and hence is not zero. Hence, must $\mu_1 = \mu_2 = \ldots = \mu_m = 0$.

Introducing the linearly independent functions $\eta_1, \ldots, \eta_m$ as new indices the substitution S takes the canonical form

$$S': \; \eta_i' = K_i\eta_i. \qquad (i = 1, \ldots, m)$$

If we take a suitable multiple of η_1 in place of η_1, we may suppose the reduction of S to S' to be accomplished by a transformation of indices of determinant unity.

In general, however, the roots of $\Delta(K) = 0$ are not all distinct. Let

$$\Delta(K) \equiv [F_k(K)]^{\alpha}[F_l(K)]^{\beta} \ldots, \qquad (m = k\alpha + l\beta + \ldots)$$

where $F_k(K), F_l(K), \ldots$ are the distinct factors of $\Delta(K)$ which belong to and are irreducible in the $GF[p^n]$. For the roots of $F_k(K)=0$ we employ the notation

$$K_0,\ K_1 \equiv K_0^{p^n},\quad K_2 \equiv K_0^{p^{2n}},\ \ldots\ldots,\quad K_{k-1} \equiv K_0^{p^{n(k-1)}}.$$

Likewise, the roots of $F_l(K)=0$ are designated

$$L_0,\ L_1 \equiv L_0^{p^n},\quad L_2 \equiv L_0^{p^{2n}},\ \ldots\ldots,\quad L_{l-1} \equiv L_0^{p^{n(l-1)}}.$$

THEOREM.—*By a suitable transformation of indices, S can be reduced to a canonical form of the following type:*

$$\begin{array}{llll}
\eta'_{i1} & = K_i\eta_{i1}, & \eta'_{ij} & = K_i(\eta_{ij}+\eta_{ij-1}), \\
 & & (j=2,\ldots\ldots,a_1) & \\
\eta'_{ia_1+1} & = K_i\eta_{ia_1+1}, & \eta'_{ia_1+j} & = K_i(\eta_{ia_1+j}+\eta_{ia_1+j-1}), \\
 & & (j=2,\ldots\ldots,a_2) & \\
\eta'_{ia_1+a_2+1} & = K_i\eta_{ia_1+a_2+1}, & \eta'_{ia_1+a_2+j} & = K_i(\eta_{ia_1+a_2+j}+\eta_{ia_1+a_2+j-1}), \\
 & & (j=2,\ldots\ldots,a_3) & \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\multicolumn{4}{c}{(i=0,1,\ldots\ldots,k-1)} \\
\zeta'_{i1} & = L_i\zeta_{i1}, & \zeta'_{ij} & = L_i(\zeta_{ij}+\zeta_{ij-1}), \\
 & & (j=2,\ldots\ldots,b_1) & \\
\zeta'_{ib_1+1} & = L_i\zeta_{ib_1+1}; & \zeta'_{ib_1+j} & = L_i(\zeta_{ib_1+j}+\zeta_{ib_1+j-1}), \\
 & & (j=2,\ldots\ldots,b_2) & \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\multicolumn{4}{c}{(i=0,1,\ldots\ldots,l-1)} \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

where $a_1+a_2+a_3+\ldots\ldots=\alpha$, $b_1+b_2+\ldots\ldots=\beta$, $\ldots\ldots$, and where the indices have the properties:

(*a*). *The indices η_{0s} $(s=1,\ldots\ldots,\alpha)$ are linear homogeneous functions of the original indices ξ_i having as coefficients polynomials in K_0 with coefficients in the $GF[p^n]$. The η_{is} are the conjugate functions derived upon replacing K_0 by K_i.*

(*b*). *The indices ζ_{0s} $(s=1,\ldots\ldots,\beta)$ are linear functions of the ξ_i having as coefficients polynomials in L_0 with coefficients in the $GF[p^n]$. The ζ_{is} are obtained from the ζ_{0s} upon replacing L_0 by L_i.*

(*c*). *The $k\alpha$ indices η_{is} $(i=0,1,\ldots\ldots,k-1;\ s=1,\ldots\ldots,\alpha)$ may be replaced by $k\alpha$ linear functions y_{is} of the indices ξ_i with coefficients in the $GF[p^n]$, such that S replaces each y_{is} by a linear function of the y's alone with coefficients in the field.*

(*d*). *The* $l\beta$ *indices* ζ_{is} *may be replaced by an equal number of linear functions* z_{is} *belonging to the* $GF[p^n]$, *such that* S *replaces each by a linear function of the* z*'s alone with coefficients in the field, etc.*

For the case $\alpha = \beta = \ldots\ldots = 1$, we obtained above the canonical form

$$\eta'_{i1} = K_i\eta_{i1}\ (i = 0, 1, \ldots\ldots, k-1), \quad \zeta'_{i1} = L_i\zeta_{i1}\ (i = 0, 1, \ldots\ldots, l-1), \ldots\ldots,$$

where η_{i1} is the same function of the ξ_j's and K_i that η_{01} is of the ξ_j's and K_0, and similarly for the ζ_{i1} as functions of the L_i.

We will prove the general theorem by induction, supposing it true for every substitution belonging to the $GF[p^n]$ whose characteristic determinant has no irreducible factors other than $F_k(K)$, $F_l(K)$, ..., and has these to a degree at most $\alpha - 1$, β, ... respectively. We will prove that the theorem is true for any substitution S for which these factors occur to the degree α, β, ... respectively, where $\alpha > 1$.

Corresponding to the distinct roots $K_0, K_1, \ldots\ldots, K_{k-1}$ of $F_k(K) = 0$, we obtain as above a set of linearly independent conjugate functions $\lambda_0, \lambda_1, \ldots\ldots, \lambda_{k-1}$ which S multiplies by $K_0, K_0, \ldots\ldots, K_{k-1}$ respectively. We may introduce these in place of an equal number of the original indices, e. g., $\xi_{m-k+1}, \ldots\ldots, \xi_m$. The substitution S then takes the form

$$S' \begin{cases} \lambda'_i = K_i\lambda_i, & (i = 0, 1, \ldots\ldots, k-1) \\ \xi'_i = \sum_{j=1}^{m} \beta_{ij}\xi_j + \sum_{j=0}^{k-1} \gamma_{ij}\lambda_j. & (i = 1, \quad \ldots\ldots, m-k) \end{cases}$$

The coefficients β_{ij} *belong to the* $GF[p^n]$. Indeed, we may set

$$\lambda_i \equiv X_0 + K_iX_1 + K_i^2X_2 + \ldots\ldots + K_i^{k-1}X_{k-1}, \\ (i = 0, 1, \ldots\ldots, k-1)$$

where the X_i are linear functions of the ξ_i with coefficients in the $GF[p^n]$. Since the λ_i are linearly independent, the X_i must be independent. Since

$$|K_i^j| \equiv \prod_{r<s}^{0,\ldots,k-1} (K_r - K_s) \neq 0,$$

the X_j can be expressed as linear functions of the λ_i. Taking the X_j as new indices in place of the λ_i, S' takes the form S'', a substitution on the indices X_j

and ξ_i with coefficients in the $GF[p^n]$. But S'' replaces ξ_i by

$$\sum_{j=1}^{m-k} \beta_{ij}\xi_j + \sum_{j=0}^{k-1} \delta_{ij} X_j .$$

for $i = 1, \ldots, m-k$. Since these functions belong to the field for arbitrary ξ_j and X_j, the coefficients β_{ij}, δ_{ij} must belong to the field.

Since the determinant of a linear substitution is not altered by a linear transformation of indices, the determinant of S' equals the determinant of S,

$$K_0 K_1 \ldots K_{k-1} \cdot |\beta_{ij}| = D.$$

We may, therefore, consider the following substitution in the $GF[p^n]$:

$$S_1: \quad \xi_i' = \sum_{j=1}^{m-k} \beta_{ij}\xi_j \qquad (i = 1, \ldots, m-k)$$

of determinant $\neq 0$. Also, the characteristic determinant $\Delta(K)$ of S equals that of the transformed substitution S', viz.:

$$\Delta(K) = \prod_{i=0}^{k-1}(K_i - K) \begin{vmatrix} \beta_{11} - K & \beta_{12} & \ldots \\ \beta_{21} & \beta_{22} - K & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}.$$

Hence, the characteristic determinant of S_1 is

$$\frac{\Delta(K)}{F_k(K)} \equiv [F_k(K)^{a-1}[F_l(K)]^{\beta} \ldots .$$

Hence, by hypothesis, S_1 can be reduced to a canonical form of the above type. Applying the same transformation of indices to S', it takes the form $\overline{S}$:

$$\lambda_i' = K_i \lambda_i, \qquad (i = 0, \ldots, k-1)$$

$$\eta_{01}' = K_0 \eta_{01} + \sum_{i=0}^{k-1} \alpha_{1i}\lambda_i, \qquad \eta_{0j}' = K_0(\eta_{0j} + \eta_{0j-1}) + \sum_{i=0}^{k-1} \alpha_{ji}\lambda_i, \qquad (j = 2, \ldots, a_1)$$

$$\eta_{0a_1+1}' = K_0 \eta_{0a_1+1} + \sum \beta_{1i}\lambda_i, \qquad \eta_{0a_1+j}' = K_0(\eta_{0a_1+j} + \eta_{0a_1+j-1}) + \sum \beta_{ji}\lambda_i, \qquad (j = 2, \ldots, a_2)$$

……

$$\zeta_{01}' = L_0 \zeta_{01} + \sum \alpha_{1i}'\lambda_i, \qquad \zeta_{0j}' = L_0(\zeta_{0j} + \zeta_{0j-1}) + \sum \alpha_{ji}'\lambda_i, \qquad (j = 2, \ldots, b_1).$$

……

the expression for η'_{is} being derived from that for η'_{0s} by replacing K_0 by K_i; the expression for ζ'_{is} from ζ'_{0s} upon replacing L_0 by L_i, etc.

To simplify the form of $\bar{S}$, introduce as new indices

$$Y_{0s} \equiv \eta_{0s} + \sum_{i=0}^{k-1} A_{si} \lambda_i \ (s=1, \ldots, \alpha); \quad Z_{0s} \equiv \zeta_{0s} + \sum_{i=0}^{k-1} B_{si} \lambda_i, \ (s=1, \ldots, \beta); \ldots$$

and the conjugate functions Y_{is}, Z_{is}, ... Then $\bar{S}$ replaces Y_{01}, Y_{02}, Y_{03} by

$$K_0 \eta_{01} + \sum_{i=0}^{k-1} (\alpha_{1i} \lambda_i + K_i A_{1i} \lambda_i)$$
$$\equiv K_0 Y_{01} + \alpha_{10} \lambda_0 + \sum_{i=1}^{k-1} [\alpha_{1i} + (K_i - K_0) A_{1i}] \lambda_i,$$
$$K_0 (\eta_{02} + \eta_{01}) + \sum (\alpha_{2i} \lambda_i + K_i A_{2i} \lambda_i)$$
$$\equiv K_0 (Y_{02} + Y_{01}) + \sum_{i=0}^{k-1} [\alpha_{2i} - K_0 A_{1i} + (K_i - K_0) A_{2i}] \lambda_i,$$
$$K_0 (\eta_{03} + \eta_{02}) + \sum (\alpha_{3i} \lambda_i + K_i A_{3i} \lambda_i)$$
$$\equiv K_0 (Y_{03} + Y_{02}) + \sum_{i=0}^{k-1} [\alpha_{3i} - K_0 A_{2i} + (K_i - K_0) A_{3i}] \lambda_i.$$

Each term of the sums on the right may be made zero by choice of the A_{ji}, viz., the terms of the first sum by choice of $A_{11}, \ldots, A_{1\,k-1}$; those of the second sum by choice of $A_{10}, A_{21}, \ldots, A_{2\,k-1}$; those of the third sum by choice of $A_{20}, A_{31}, \ldots, A_{3\,k-1}$. A like result holds for all of the Y_{0s} $(s=1, \ldots, \alpha)$.

$\bar{S}$ replaces Z_{01}, Z_{02}, ... by respectively

$$L_0 Z_{01} + \sum_{i=0}^{k-1} [\alpha'_{1i} + (K_i - L_0) B_{1i}] \lambda_i,$$
$$L_0 (Z_{02} + Z_{01}) + \sum_{i=0}^{k-1} [\alpha'_{2i} - L_0 B_{1i} + (K_i - L_0) B_{2i}] \lambda_i, \ \ldots.$$

Since $K_i - L_0 \neq 0$, the coefficients of λ_i may be made to vanish by choice of the B_{si}. Hence, $\bar{S}$ takes the form S_2:

$$\lambda'_i = K_i \lambda_i,$$

$$Y'_{i1} = K_i Y_{i1} + \phi(K_i)\lambda_i, \qquad Y'_{ij} = K_i(Y_{ij} + Y_{ij-1}),$$
$$(j = 2, \ldots, a_1)$$
$$Y'_{ia_1+1} = K_i Y_{ia_1+1} + \psi(K_i)\lambda_i, \qquad Y'_{ia_1+j} = K_i(Y_{ia_1+j} + Y_{ia_1+j-1}),$$
$$(j = 2, \ldots, a_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(i = 0, 1, \ldots, k-1)$$
$$Z'_{i1} = L_i Z_{i1}, \qquad Z'_{ij} = L_i(Z_{ij} + Z_{ij-1}),$$
$$(j = 2, \ldots, b_1)$$
$$Z'_{ib_1+1} = L_i Z_{ib_1+1}, \qquad Z'_{ib_1+j} = L_i(Z_{ib_1+j} + Z_{ib_1+j-1}).$$
$$(j = 2, \ldots, b_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(i = 0, 1, \ldots, l-1)$$

If the constants $\phi(K_i)$, $\psi(K_i)$, $\chi(K_i)$, ... are all zero, no further reduction is necessary. If any two are not zero, as ϕ and ψ, suppose for definiteness that $a_1 \overset{=}{<} a_2$, and introduce in place of $Y_{i1}, \ldots, Y_{ia_1}$ the new indices

$$\overline{Y}_{ij} \equiv Y_{ij} - \frac{\phi}{\psi} Y_{ia_1+j}. \qquad (i = 1, \ldots, a_1).$$

The substitution S_2 replaces $\overline{Y}_{i1}$, $\overline{Y}_{ij}$ $(j = 2, \ldots, a_1)$ by respectively

$$K_i \overline{Y}_{i1}, \quad K_i(\overline{Y}_{ij} + \overline{Y}_{ij-1}). \qquad (j = 2, \ldots, a_1)$$

Hence, the introduction of the $\overline{Y}_{ij}$ has the effect of setting $\phi = 0$ in S_2. Proceeding similarly, we can suppose that ϕ, ψ, χ, ... are all zero but one, say $\psi \neq 0$. In the latter case, we set

$$\psi(K_i) . \lambda_i \equiv K_i y_{ia_1}$$

and find for S_2 the canonical form

$$\begin{aligned}
Y'_{i1} &= K_i Y_{i1}, & Y'_{ij} &= K_i(Y_{ij} + Y_{ij-1}), & &(j = 2, \ldots, a_1)\\
y'_{ia_1} &= K_i y_{i1a}, & Y'_{ia_1+1} &= K_i(Y_{ia_1+1} + y_{ia_1}), & &\\
& & Y'_{ia_1+j} &= K_i(Y_{ia_1+j} + Y_{ia_1+j-1}), & &(j = 2, \ldots, a_2)\\
Y'_{ia_1+a_2+1} &= K_i Y_{ia_1+a_2+1}, & Y'_{ia_1+a_2+j} &= K_i(Y_{ia_1+a_2+j} + Y_{ia_1+a_2+j-1}), & &\\
& & & & &(j = 2, \ldots, a_3)
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Z'_{i1} = L_i Z_{i1}, \qquad Z'_{ij} = L_i(Z_{ij} + Z_{ij-1}), \qquad (j = 2, \ldots, b_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

In every case we reach a canonical form of the type given in the theorem, for which the indices Y_{is} have the properties (a). But the indices Z_{ij} are linear functions of the ξ_i with coefficients which certainly involve L_i and apparently[*] also K_i. If the K_i be involved, we proceed as follows: From the canonical form actually reached, $S = YS_1$, where Y is the partial substitution on the indices Y_{ij} not altering the indices Z_{ij}, etc., while S_1 does not involve the indices Y_{ij}, but affects the Z_{ij}, etc. Setting

$$Y_{is} \equiv y_s + y'_s K_i + y''_s K_i^2 + y_s^{(k-1)} K_i^{k-1},$$
$$(s = 1, \ldots, \alpha;\ i = 0, \ldots, k-1)$$

where the y's are linear functions of the ξ_i with coefficients in the $GF[p^n]$, we can evidently introduce the y's as new indices in place of the Y_{is}, so that Y takes the form of a substitution belonging to the $GF[p^n]$ and affecting only $k\alpha$ indices. Likewise, by introducing in place of the Z_{ij}, etc., an equal number of linear functions z_{ij}, etc., belonging to the $GF[p^n]$, it is possible to give to S_1 the form of a substitution in the field and affecting only $m - k\alpha$ indices. Its characteristic determinant is $[F_i(K)]^{\beta} \ldots$ Hence, by the hypothesis made for the induction, S_1 can be reduced by a linear transformation T to a canonical form

$$\zeta'_{i1} = L_i \zeta_{i1}, \quad \zeta'_{ij} = L_i(\zeta_{i1} + \zeta_{i\,j-1}), \qquad (j = 2, \ldots, b_1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where the ζ_{ij} are linear functions of the ζ_i with coefficients involving the imaginary L_i only. As the transformation T does not alter the indices which Y affects, we obtain the desired canonical form.

3. Consider as an example the substitution in the $GF[p^n]$, p^n of the form $4l - 1$,

$$S: \quad \xi'_1 = -2\xi_2 - \xi_4, \quad \xi'_2 = \xi_1, \quad \xi'_3 = \xi_2, \quad \xi'_4 = \xi_3,$$

having the characteristic determinant

$$\Delta(K) \equiv (K^2 + 1)^2,$$

where $K^2 + 1$ is irreducible in the field. A root of $i^2 = -1$ belongs to the $GF[p^{2n}]$ but not to the $GF[p^n]$. The functions which S multiplies by i and

---

[*] By the considerations in the text, we may dispense with the difficult proof, analogous to that of Jordan, "Traité," pp. 121-122, that the Z_{ij} do not involve K, but the single imaginary L_i.

$-i$ are readily found to be respectively

$$\lambda_1 \equiv -ix_1 + x_2 - ix_3 + x_4, \quad \lambda_2 \equiv ix_1 + x_2 + ix_3 + x_4.$$

Introducing λ_1, λ_2 in place of the indices x_2, x_3, S takes the form

$$x_1' = x_4 - \lambda_1 - \lambda_2, \quad x_4' = -x_1 + i/2\,\lambda_1 - i/2\,\lambda_2, \quad \lambda_1' = i\lambda_1, \quad \lambda_2' = -i\lambda_2.$$

The partial substitution of determinant unity,

$$x_1' = x_4, \quad x_4' = -x_1$$

multiplies $y_1 \equiv x_1 - ix_4$ by i and multiplies $y_2 \equiv x_1 + ix_4$ by $-i$. Introducing y_1 and y_2 as new indices in place of x_1 and x_4, S takes the form

$$y_1' = iy_1 - \tfrac{1}{2}\lambda_1 - \tfrac{3}{2}\lambda_2, \quad y_2' = -iy_2 - \tfrac{1}{2}\lambda_2 - \tfrac{3}{2}\lambda_1, \quad \lambda_1' = i\lambda_1, \quad \lambda_2' = -i\lambda_2.$$

Introducing as new indices,

$$\overline{\lambda}_1 \equiv i/2\,\lambda_1, \quad \overline{\lambda}_2 \equiv -i/2\,\lambda_2, \quad \overline{y}_1 \equiv y_1 + \frac{3i}{4}\lambda_2, \quad \overline{y}_2 \equiv y_2 - \frac{3i}{4}\lambda_1,$$

S takes the canonical form

$$\overline{\lambda}_1' = i\overline{\lambda}_1, \quad \overline{y}_1' = i(\overline{y}_1 + \overline{\lambda}_1), \quad \overline{\lambda}_2' = -i\overline{\lambda}_2, \quad \overline{y}_2' = -i(\overline{y}_2 + \overline{\lambda}_2),$$

where $\overline{\lambda}_1$ and $\overline{\lambda}_2$ are conjugate linear functions of $\xi_1, \xi_2, \xi_3, \xi_4$, and likewise for $\overline{y}_1$, $\overline{y}_2$.

Substitutions commutative with a given substitution S, §§4–7.

4. From the canonical form of the general m-ary linear homogeneous substitution S in the $GF[p^n]$, it follows that S may be expressed as a product

$$S \equiv Y_0\, Y_1 \ldots . \, Y_{k-1} \; Z_0 Z_1 \ldots . \, Z_{l-1} \ldots . ,$$

where, for $i = 0, 1, \ldots . , k-1$, Y_i denotes the substitution on $a_1 + a_2 + \ldots . + a_{r+1} \equiv \alpha$ indices:

$$\begin{array}{llr}
\eta_{i1}' = K_i \eta_{i1}, & \eta_{ij}' = K_i(\eta_{ij} + \eta_{i\,j-1}), & (j = 2, \ldots . , a_1) \\
\eta_{i\,a_1+1}' = K_i \eta_{i\,a_1+1}', & \eta_{i\,a_1+j}' = K_i(\eta_{i\,a_1+j} + \eta_{i\,a_1+j-1}), & (j = 2, \ldots . , a_2) \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots & \\
\eta_{i\,a_1+\ldots.+a_r+1}' = K_i \eta_{i\,a_1+\ldots.+a_r+1}, & & \\
\eta_{i\,a_1+\ldots.+a_r+j}' = K_i(\eta_{i\,a_1+\ldots.+a_r+j} + \eta_{i\,a_1+\ldots.+a_r+j-1}), & & (j = 2, \ldots . , a_{r+1})
\end{array}$$

while, for $i = 0, 1, \ldots, l-1$, Z_i denotes the substitution on $b_1 + b_2 + \ldots + b_{s+1} \equiv \beta$ indices:

$$\begin{array}{lll} \zeta'_{i1} = L_i\zeta_{i1}, & \zeta'_{ij} = L_i(\zeta_{ij} + \zeta_{ij-1}), & (j = 2, \ldots, b_1). \\ \zeta'_{i\,b_1+1} = L_i\zeta_{i\,b_1+1}, & \zeta'_{i\,b_1+j} = L_i(\zeta_{i\,b_1+j} + \zeta_{i\,b_1+j-1}), & (j = 2, \ldots, b_2) \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots & \end{array}$$

where in each case are written only those indices which the substitution alters.

Let B be a second m-ary linear homogeneous substitution in the $GF[p^n]$, and suppose it to be expressed in terms of the above indices $\eta_{is}, \zeta_{is}, \ldots$, by means of which the given substitution S is reduced to the above canonical form.

THEOREM.—*In order that B shall be commutative with S, it is necessary and sufficient that the following two conditions be satisfied:*

(1). *B shall break up into the product*

$$B = y_0 y_1 \ldots y_{k-1}\, z_0 z_1 \ldots z_{l-1} \ldots,$$

where y_0 is a substitution of determinant $\neq 0$ altering only the indices $\eta_{01}, \ldots, \eta_{0\alpha}$:

$$\eta'_{0s} = \sum_{j=1}^{\alpha} \beta_{sj}(K_0)\,\eta_{0j}, \qquad (s = 1, \ldots, \alpha)$$

the coefficients $\beta_{sj}(K_0)$ being polynomials in K_0 with coefficients in the $GF[p^n]$, and for $i = 0, 1, \ldots, k-1$, y_i denotes the conjugate substitution

$$\eta'_{is} = \sum_{j=1}^{\alpha} \beta_{sj}(K_i)\,\eta_{ij}, \qquad (s = 1, \ldots, \alpha)$$

while, similarly, z_i (for $i = 0, 1, \ldots, l-1$) denotes the substitution of determinant $\neq 0$ altering only $\zeta_{i1}, \ldots, \zeta_{i\beta}$, viz.,

$$\zeta'_{is} = \sum_{j=1}^{\beta} \gamma_{sj}(L_i)\,\zeta_{ij}. \qquad (s = 1, \ldots, \beta)$$

(2). *Y_0 and y_0 shall be commutative, Z_0 and z_0 shall be commutative, etc.*

The method of proof is sufficiently illustrated by the developments of Jordan ("Traité," pp. 128–132) upon a particular example.

5. As an important example, suppose that S has the canonical form

$$\begin{array}{ll} \eta'_i = K_i\eta_i. & (i = 0, 1, \ldots, k-1) \\ \zeta'_i = L_i\zeta_i, & (i = 0, 1, \ldots, l-1), \ldots, \\ \xi'_i = R_i\xi_i. & (i = 0, 1, \ldots, r-1) \end{array}$$

Then B has the form

$$\begin{array}{ll} \eta_i' = \varkappa(K_i)\,\eta_i, & (i = 0,\ 1, \ldots, k-1) \\ \zeta_i' = \lambda(L_i)\,\zeta_i, & (i = 0,\ 1, \ldots, l-1), \ldots, \\ \xi_i' = \rho(R_i)\,\xi_i. & (i = 0,\ 1, \ldots, r-1) \end{array}$$

If $K, L, \ldots, R$ be primitive roots of the Galois fields of orders $p^{nk}, p^{nl}, \ldots, p^{nr}$ respectively, we may set

$$\varkappa(K_0) = K^a, \quad \lambda(L_0) = L^b, \ldots, \quad \rho(R_0) = R^c.$$

Then B takes the form

$$\begin{array}{ll} \eta_i' = K^{ap^{ni}}\eta_i, & (i = 0, \ldots, k-1) \\ \zeta_i' = L^{bp^{ni}}\zeta_i, & (i = 0, \ldots, l-1), \ldots, \\ \xi_i' = R^{cp^{ni}}\xi_i. & (i = 0, \ldots, r-1) \end{array}$$

6. While investigating the conditions under which Y_0 is commutative with y_0, we write Y_0 in the form (where we set $\eta_j \equiv \eta_{0j}$):

$$Y_0 \quad \begin{cases} \eta_j' = K_0\,\eta_j, & j = 1,\ a_1+1,\ a_1+a_2+1, \ldots, a_1 + \ldots + a_r + 1) \\ \eta_j' = K_0(\eta_j + \eta_{j-1}). & j = 2, \ldots, \alpha;\ j \neq a_1+1, \ldots, a_1 + \ldots + a_r + 1) \end{cases}$$

The most general form possible for y_0 is

$$y_0\colon\ \eta_i' = \sum_{j=1}^{\alpha} \alpha_{ij}\,\eta_j. \qquad (i = 1, \ldots, \alpha)$$

Equating the expressions by which $y_0\,Y_0$ and $Y_0\,y_0$ replace η_i, where $i = 1,\ a_1+1,\ a_1+a_2+1, \ldots, a_1 + \ldots + a_r + 1$, in succession, we find the conditions

$$\alpha_{ij} = 0. \quad \begin{pmatrix} j = 2, \ldots, \alpha;\ j \neq a_1+1,\ a_1+a_2+1, \ldots, a_1 + \ldots + a_r + 1 \\ i = 1,\ a_1+1,\ a_1+a_2+1, \ldots, a_1 + \ldots + a_r + 1. \end{pmatrix} \tag{2}$$

Equating the expressions by which they replace η_i, where, in succession, $i = 1, \ldots, \alpha;\ i \neq 1,\ a_1+1, \ldots, a_1 + \ldots + a_r + 1$, we obtain the condition

$$\sum_{j \neq a_1+1, \ldots, a_1 + \ldots + a_r + 1}^{j=2, \ldots, \alpha} \alpha_{ij}\,\eta_{j-1} = \sum_{j=1}^{\alpha} \alpha_{i-1\,j}\,\eta_j \equiv \sum_{j=2}^{\alpha+1} \alpha_{i-1\,j-1}\,\eta_{j-1}.$$

Hence, for $i = 2, \ldots, \alpha;\ i \neq a_1 + 1,\ a_1 + a_2 + 1, \ldots,\ a_1 + a_2 + \ldots + a_r + 1,$

$$\alpha_{i-1\,j-1} = 0, \quad (j = a_1 + 1,\ a_1 + a_2 + 1, \ldots, a_1 + \ldots + a_r + 1,\ a + 1) \quad (3)$$

$$\alpha_{i-1\,j-1} = \alpha_{ij}, \quad (j = 2, \ldots, a;\ j \neq a_1 + 1,\ a_1 + a_2 + 1, \ldots, a_1 + \ldots + a_r + 1) \quad (4)$$

By way of example, suppose $r = 2$ and that $a_1 = a_2 = 4,\ a_3 = 2$. Then by (2), (3), (4),

$$\begin{aligned} &\alpha_{1j} = \alpha_{5j} = \alpha_{9j} = 0, && (j = 2, 3, 4, 6, 7, 8, 10) \\ &\alpha_{j4} = \alpha_{j8} = \alpha_{j10} = 0, && (j = 1, 2, 3, 5, 6, 7, 9) \\ &\alpha_{ij} = \alpha_{i-1\,j-1}. && (i, j = 2, 3, 4, 6, 7, 8, 10). \end{aligned}$$

Hence y_0 takes the form

$$\begin{array}{r|cccc|cccc|cc|}
 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 & \eta_{10} \\
\hline
\eta_1' = & \alpha_{11} & 0 & 0 & 0 & \alpha_{15} & 0 & 0 & 0 & 0 & 0 \\
\eta_2' = & \alpha_{21} & \alpha_{11} & 0 & 0 & \alpha_{25} & \alpha_{15} & 0 & 0 & 0 & 0 \\
\eta_3' = & \alpha_{31} & \alpha_{21} & \alpha_{11} & 0 & \alpha_{35} & \alpha_{25} & \alpha_{15} & 0 & \alpha_{39} & 0 \\
\eta_4' = & \alpha_{41} & \alpha_{31} & \alpha_{21} & \alpha_{11} & \alpha_{45} & \alpha_{35} & \alpha_{25} & \alpha_{15} & \alpha_{49} & \alpha_{39} \\
\hline
\eta_5' = & \alpha_{51} & 0 & 0 & 0 & \alpha_{55} & 0 & 0 & 0 & 0 & 0 \\
\eta_6' = & \alpha_{61} & \alpha_{51} & 0 & 0 & \alpha_{65} & \alpha_{55} & 0 & 0 & 0 & 0 \\
\eta_7' = & \alpha_{71} & \alpha_{61} & \alpha_{51} & 0 & \alpha_{75} & \alpha_{65} & \alpha_{55} & 0 & \alpha_{79} & 0 \\
\eta_8' = & \alpha_{81} & \alpha_{71} & \alpha_{61} & \alpha_{51} & \alpha_{85} & \alpha_{75} & \alpha_{65} & \alpha_{55} & \alpha_{89} & \alpha_{79} \\
\hline
\eta_9' = & \alpha_{91} & 0 & 0 & 0 & \alpha_{95} & 0 & 0 & 0 & \alpha_{99} & 0 \\
\eta_{10}' = & \alpha_{10\,1} & \alpha_{91} & 0 & 0 & \alpha_{10\,5} & \alpha_{95} & 0 & 0 & \alpha_{10\,9} & \alpha_{99} \\
\hline
\end{array}$$

By simple interchange of rows and columns, we see that the determinant of the above substitution on ten indices equals

$$\alpha_{99}^2 \begin{vmatrix} \alpha_{11} & \alpha_{15} \\ \alpha_{51} & \alpha_{55} \end{vmatrix}^4 .$$

7. For the general case, the matrix of y_0 may be considered to be made up of $(r+1)^2$ rectangles, the general one of which R_{ij} includes $a_i\,a_j$ coefficients α_{rs}. If $a_i \leqq a_j$, this rectangle includes a square array S_i of coefficients, a_i to a side. The coefficients in its diagonal are all equal, likewise those in any parallel to the diagonal. All coefficients of the rectangle R_{ij} which lie above or to the right of the diagonal of the square S_i are zeros. The total number of distinct coefficients

$\neq 0$ is, therefore, (supposing $a_1 \geqq a_2 \geqq a_3 \geqq \ldots .$),

$$
\begin{aligned}
&(a_1 + a_2 + a_3 + \ldots . + a_{r+1}) + (a_2 + a_2 + a_3 + \ldots . + a_{r+1}) \\
&\quad + (a_3 + a_3 + a_3 + \ldots . + a_{r+1}) + \ldots . + (a_{r+1} + a_{r+1} + a_{r+1} + \ldots . + a_{r+1}) \\
&\qquad \equiv a_1 + 3a_2 + 5a_3 + 7a_4 + \ldots . + (2k-1)\,a_k + \ldots . + (2r+1)\,a_{r+1}, \quad (5),
\end{aligned}
$$

the quantities in the j^{th} parenthesis being the number of distinct coefficients $\neq 0$ in the j^{th} row of rectangles.

To fix the ideas, let

$$a_1 = a_2 = \ldots = a_\lambda \equiv a, \quad a_{\lambda+1} = a_{\lambda+2} = \ldots = a_{\lambda+\mu} \equiv b, \quad a_{\lambda+\mu+1} = \ldots = a_{\lambda+\mu+\rho} \equiv c,$$

where

$$a > b > c, \quad \lambda + \mu + \rho \equiv r + 1.$$

We proceed to prove that the determinant of y_0 equals

$$|y_0| = D_\lambda^a D_\mu^b D_\rho^c,$$

where

$$D_\lambda \equiv \begin{vmatrix} \alpha_{11} & \alpha_{1\,a+1} & \alpha_{1\,2a+1} & \ldots . \alpha_{1\,\lambda a-a+1} \\ \alpha_{a+1\,1} & \alpha_{a+1\,a+1} & \alpha_{a+1\,2a+1} & \ldots . \alpha_{a+1\,\lambda a-a+1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_{\lambda a-a+1\,1} & \alpha_{\lambda a-a+1\,a+1} & \alpha_{\lambda a-a+1\,2a+1} & \ldots . \alpha_{\lambda a-a+1\,\lambda a-a+1} \end{vmatrix}.$$

$$D_\mu \equiv \begin{vmatrix} \alpha_{\lambda a+1\,\lambda a+1} & \alpha_{\lambda a+1\,\lambda a+b+1} & \ldots . \alpha_{\lambda a+1\,\lambda a+\mu b-b+1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_{\lambda a+\mu b-b+1\,\lambda a+1} & \alpha_{\lambda a+\mu b-b+1\,\lambda a+b+1} & \ldots . \alpha_{\lambda a+\mu b-b+1\,\lambda a+\mu b-b+1} \end{vmatrix}.$$

$$D_\rho \equiv \begin{vmatrix} \alpha_{\lambda a+\mu b+1\,\lambda a+\mu b+1} & \ldots . \alpha_{\lambda a+\mu b+1\,\lambda a+\mu b+\rho c-c+1} \\ \ldots\ldots & \ldots\ldots \\ \alpha_{\lambda a+\mu b+\rho c-c+1\,\lambda a+\mu b+1} & \ldots . \alpha_{\lambda a+\mu b+\rho c-c+1\,\lambda a+\mu b+\rho c-c+1} \end{vmatrix}.$$

Indeed, the coefficients of $\eta_1, \eta_{a+1}, \eta_{2a+1}, \ldots, \eta_{\lambda a+1}, \eta_{\lambda a+b+1}, \ldots, \eta_{\lambda a+\mu b+\rho c-c+1}$ in $\eta'_1, \eta'_{a+1}, \eta'_{2a+1}, \ldots .$, form a matrix of the form

$$(D) = \left(\begin{array}{c|c|c} (D_\lambda) & \text{zeros} & \text{zeros} \\ \hline & (D_\mu) & \text{zeros} \\ \hline & & (D_\rho) \end{array}\right)$$

of determinant $D = D_\lambda D_\mu D_\rho$. All the coefficients of $\eta'_1, \eta'_{a+1}, \ldots, \eta'_{\lambda a+\mu b+\rho c-c+1}$ are zeros except those in the matrix (D). Hence,

$$|y_0| = D_\lambda D_\mu D_\rho . D_1,$$

D_1 being the determinant of the matrix (D_1) obtained from the matrix for y_0 upon deleting the 1st, $a+1$st, $2a+1$st, . . . ; $\lambda a+\mu b+\rho c-c+1$st rows and columns. Selecting from (D_1) the coefficients of $\eta_2, \eta_{a+2}, \eta_{2a+2}, \ldots$ in $\eta'_2, \eta'_{a+2}, \ldots$, we obtain the matrix (D). Deleting the corresponding rows and columns from (D_1), we obtain a matrix (D_2), whose determinant D_2 is given by

$$D_1 = DD_2.$$

After c such operations, we reach a matrix (D_c) whose determinant D_c is given by the equation

$$|y_0| = D^c D_c.$$

In the matrix (D_c), the coefficients of $\eta_{c+1}, \eta_{a+c+1}, \ldots, \eta_{\lambda a+c+1}, \eta_{\lambda a+b+c+1}, \ldots, \eta_{\lambda a+\mu b-b+c+1}$ in $\eta'_{c+1}, \eta'_{a+c+1}, \ldots$, form a matrix

$$(E) \equiv \left(\begin{array}{c|c} (D_\lambda) & \text{zeros} \\ \hline & (D_\mu) \end{array}\right)$$

of determinant $E = D_\lambda D_\mu$. In (D_c) all the coefficients of $\eta'_{c+1}, \eta'_{a+c+1}, \ldots$ are zero except those in (E). Hence,

$$D_c = E \,.\, E_1,$$

where E_1 is the determinant of the matrix (E_1) obtained by deleting the 1st, $a+1$st, $2a+1$st, ... , $\lambda a+\mu b-b+1$st rows and columns from (D_c). Selecting the coefficients of $\eta_{c+2}, \eta_{a+c+2}, \ldots$ in $\eta'_{c+2}, \ldots$ from the matrix (E_1), we obtain the matrix (E). Deleting from (E_1) the 2nd, $a+2$nd, ... rows and columns, we obtain a matrix (E_2), whose determinant E_2 is given by the equation $E_1 = EE_2$. After $b-c$ such steps, we reach a matrix (E_{b-c}) whose determinant is given by

$$D_c = E^{b-c} E_{b-c}.$$

In the matrix (E_{b-c}), the coefficients of $\eta_{b+i}, \eta_{a+b+i}, \ldots, \eta_{\lambda a-a+b+i}$ in $\eta'_{b+i}, \eta'_{a+b+i}, \ldots, \eta'_{\lambda a-a+b+i}$ (for each value of $i = 1, 2, \ldots, a-b$), form the matrix (D_λ). Hence,

$$E_{b-c} = D_\lambda^{a-b}.$$

Hence, $$|y_0| = (D_\lambda D_\mu D_\rho)^c (D_\lambda D_\mu)^{b-c} D_\lambda^{a-b} \equiv D_\lambda^a D_\mu^b D_\rho^c.$$

There are $\lambda^2 + \mu^2 + \rho^2$ distinct coefficients entering the determinants D_λ, D_μ, D_ρ. These are subject to the conditions that the determinants shall not vanish.

By (5), the total number of non-vanishing coefficients of y_0 is seen to be

$$a\lambda^2 + b\,(\mu^2 + 2\lambda\mu) + c\,(\rho^2 + 2\lambda\rho + 2\mu\rho).$$

The number of absolutely arbitrary coefficients in y_0 is, therefore,

$$\lambda^2\,(a-1) + \mu^2\,(b-1) + \rho^2\,(c-1) + 2b\lambda\mu + 2c\lambda\rho + 2c\mu\rho.$$

To formulate our theorem for the most general case, let

$$a_1 = a_2 = \ldots . = a_{\lambda_1} \equiv A_1; \quad a_{\lambda_1+1} = \ldots . = a_{\lambda_1+\lambda_2} \equiv A_2; \ldots .;$$
$$a_{\lambda_1+\lambda_2+\ldots .+\lambda_{s-1}+1} = \ldots . = a_{\lambda_1+\lambda_2+\ldots .+\lambda_s} \equiv A_s,$$

where $\qquad A_1 > A_1 > \ldots . > A_s; \quad \lambda_1 + \lambda_2 + \ldots . + \lambda_s = r + 1.$

The determinant of y_0 equals

$$D_{\lambda_1}^{A_1}\,D_{\lambda_2}^{A_2} \ldots . D_{\lambda_s}^{A_s}.$$

The coefficients α_{ij} being functions of K_0, a root of an irreducible equation of degree k belonging to the $GF\,[p^n]$, the number of sets of values for the λ_i^2 coefficients entering D_λ for which this determinant $\neq 0$ is*

$$N_i \equiv (p^{nk\lambda_i} - 1)(p^{nk\lambda_i} - p^{nk}) \ldots . (p^{nk\lambda_i} - p^{nk(\lambda_i - 1)}).$$

Hence, there are $N_1 N_2 \ldots . N_s$ sets of values for the $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \ldots . + \lambda_s^2$ distinct coefficients entering these determinants. But by (5), the number of distinct coefficients in y_0 is

$$A_1\lambda_1^2 + A_2\lambda_2\,(\lambda_2 + 2\lambda_1) + A_3\lambda_3\,(\lambda_3 + 2\lambda_2 + 2\lambda_1) + \ldots .$$
$$+ A_s\lambda_s\,(\lambda_s + 2\lambda_{s-1} + \ldots . + 2\lambda_1).$$

There remain the following number of wholly arbitrary coefficients:

$$A \equiv \sum_{i=1}^{s} \lambda_i^2 (A_i - 1) + 2A_2\lambda_2\lambda_1 + 2A_3\lambda_3(\lambda_2 + \lambda_1) + \ldots .$$
$$+ 2A_s\lambda_s\,(\lambda_{s-1} + \ldots . + \lambda_2 + \lambda_1).$$

Each of these has p^{nk} distinct values. The total number of substitutions y_0 commutative with Y_0 is, therefore,

$$N_1 N_2 \ldots . N_s . p^{nkA}. \tag{6}$$

For the above example, viz., $a_1 = a_2 = a_4$, $a_3 = 2$, this number is

$$p^{21nk}\,(p^{2nk} - 1)(p^{2nk} - p^{nk})(p^{nk} - 1),$$

as is evident by inspecting the form of y_0.

---

* Jordan, "Traité," p. 185; Dickson, Annals of Mathematics, 1897, p. 164.

8. THEOREM.—*Two real linear homogeneous substitutions S and T on the indices $\xi_1, \ldots, \xi_m$ have the same canonical form C if, and only if, T is the transformed of S by a real linear homogeneous substitution W on the same indices.*[*]

If $T = W^{-1}SW$, S can be reduced to T by the introduction of new indices defined by the real transformation W. Hence, S and T have the same canonical form.

Suppose, inversely, that two real substitutions S and T on the indices ξ_i can be reduced to the same canonical form by transformations $[S]$ and $[T]$ respectively. Let $[T]$ denote the transformation from the indices $\xi_1, \ldots, \xi_m$ to the indices $\eta_{is}, \zeta_{is}, \ldots$, where

$$\eta_{is} \equiv Y_s + Y_s' K_i + Y_s'' K_i^2 + \ldots + Y_s^{(k-1)} K_i^{k-1},$$
$$(s = 1, \ldots, \alpha;\ i = 0, 1, \ldots k-1)$$
$$\zeta_{is} \equiv Z_s + Z_s' L_i + Z_s'' L_i^2 + \ldots + Z_s^{(l-1)} L_i^{l-1},$$
$$(s = 1, \ldots, \beta;\ i = 0, 1, \ldots, l-1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $Y_s, Y_s', \ldots, Z_s, \ldots$ are $k\alpha$ real linear functions of the ξ_i, which are linearly independent. Denote by τ the transformation of indices from $\eta_{is}, \zeta_{is}, \ldots$ to $Y_s, Y_s', \ldots, Z_s, \ldots$ By hypothesis, $[T]$ transforms T into the canonical form C. Let τ transform C into C_τ. Then $[T]\tau$ is a real substitution which transforms T into C_τ, a real substitution on real indices.

Similarly, let $[S]$ denote the transformation from the indices $\xi_1, \ldots, \xi_m$ to the indices $\overline{\eta}_{is}, \overline{\zeta}_{is}, \ldots$, where

$$\overline{\eta}_{is} \equiv \overline{Y}_s + \overline{Y}_s' K_i + \ldots, \quad \overline{\zeta}_{is} \equiv \overline{Z}_s + \overline{Z}_s' L_i + \ldots, \quad \ldots$$

Denote by σ the transformation of indices from $\overline{\eta}_{is}, \overline{\zeta}_{is}, \ldots$ to $\overline{Y}_s, \ldots, \overline{Z}_s, \ldots$. By hypothesis, $[S]$ transforms S into the canonical form $\overline{C}$, which is the same substitution on the indices $\overline{\eta}_{is}, \overline{\zeta}_{is}, \ldots$ that C is on the indices $\eta_{is}, \zeta_{is}, \ldots$ Let σ transform $\overline{C}$ into $\overline{C}_\sigma$. Hence, if R be the real substitution transforming $Y_s, \ldots, Z_s, \ldots$ into $\overline{Y}_s, \ldots, \overline{Z}_s, \ldots$ respectively,

$$\overline{C}_\sigma = R^{-1} C_\tau R.$$

It follows that the product

$$[T]\,\tau R\,([S]\,\sigma)^{-1}$$

is a real substitution on the indices ξ_i which transforms T into S.

---

[*] In §§8–9, we speak of a function or substitution as *real* if its coefficients belong to the $GF[p^n]$.

9. As a corollary to the theorem of §8, we have the theorem:

In order that a real substitution S whose canonical form has the multipliers $M_1, M_2, \ldots, M_m$ shall be the transformed by a real substitution of a real substitution whose canonical form has the multipliers $\Theta M_1, \Theta M_2, \ldots, \Theta M_m$, it is necessary that the two sets of multipliers be identical apart from their order. Hence, $\Theta^m = 1$.

If $m = 3$, $\Theta \neq 1$, *and if* S *have determinant unity, it is necessary that* M_1, M_2, M_3 *be in some order equal to* $1, \Theta, \Theta^2$.

Indeed, since $\Theta M_1 \neq M_1$, we may set $\Theta M_1 = M_2$, properly choosing the notation for M_2 and M_3. Then $\Theta M_2 \neq M_1$; for if so, $\Theta^2 = 1$, and hence $\Theta = 1$. Hence, $\Theta M_2 = M_3$, $\Theta M_3 = M_1$. Hence, the multipliers are

$$M_1, \quad M_2 = \Theta M_1, \quad M_3 = \Theta^2 M_1. \qquad (\Theta^3 = 1).$$

The determinant of S being unity,

$$M_1 M_2 M_3 = 1 = \Theta^3 M_1^3.$$

Hence, $M_1^3 = 1$, so that M_3 is a power of Θ. The theorem is, therefore, proven.

In like manner, we may prove that for $m = 4$, $\Theta \neq 1$ and S of determinant unity, it is necessary that the multipliers M_1, M_2, M_3, M_4 be in some order either

$$M_1, \ \Theta M_1, \ \frac{1}{M_1}, \ \frac{\Theta}{M_1}, \qquad (\Theta^2 = 1, \ M_1 \text{ arbitrary})$$

or

$$M_1, \ \Theta M_1, \ \Theta^2 M, \ \Theta^3 M_1, \quad (\Theta^2 = -1, \ M_1^4 = \Theta^2 = -1),$$

there being a single distinct set of multipliers in the latter case.

These results find constant application in the problem of the determination of the subgroups of the simple groups of all linear fractional substitutions of determinant unity in two or three non-homogeneous variables with coefficients in an arbitrary Galois field.

UNIVERSITY OF TEXAS, *December* 11, 1899.

Families of Transformations of Straight Lines into Spheres.

By E. O. Lovett.

---

1. By the transformations of the point-space (x, y, z) into the point-space (X, Y, Z), which are determined by two *æquationes directrices*

$$\Phi(x, y, z, X, Y, Z) = 0, \quad \Psi(x, y, z, X, Y, Z) = 0, \tag{1}$$

the straight line

$$y + kx + m = 0, \quad z + lx + n = 0; \tag{2}$$

is changed into the surface

$$\Omega(X, Y, Z, k, l, m, n) = 0, \tag{3}$$

whose equation is obtained by eliminating x, y, z by means of the four equations (1) and (2).

If the equations (1) are of the form

$$x\Phi_1 + y\Phi_2 + z\Phi_3 + \Phi_4 = 0, \quad x\Phi_5 + y\Phi_6 + z\Phi_7 + \Phi_8 = 0, \tag{4}$$

where the Φ_i's are any functions of X, Y, Z not containing x, y, z, the line (2) will be transformed into the surface

$$\begin{vmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_5 & \Phi_6 & \Phi_7 & \Phi_8 \\ k & 1 & 0 & m \\ l & 0 & 1 & n \end{vmatrix} = 0. \tag{5}$$

In order that this surface be a quadric for all values of k, l, m, n, it is necessary and sufficient that all the determinants of the matrix:

$$\left\| \begin{matrix} \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\ \Phi_5 & \Phi_6 & \Phi_7 & \Phi_8 \end{matrix} \right\| \tag{6}$$

reduce to functions of degree not higher than the second.

2. In particular, let all of the functions Φ_i be linear, then the equations

$$x\phi_1 + y\phi_2 + z\phi_3 + \phi_4 = 0, \quad x\phi_5 + y\phi_6 + z\phi_7 + \phi_8 = 0, \tag{7}$$

where
$$\phi_i \equiv a_i X + b_i Y + c_i Z + d_i \quad (a_i,\ b_i,\ c_i,\ d_i \text{ constants}) \tag{8}$$

define a family of ∞^{30} transformations which change the straight line (2) into the quadric

$$\begin{vmatrix} k & m \\ l & n \end{vmatrix} \cdot \begin{vmatrix} \phi_2 & \phi_3 \\ \phi_6 & \phi_7 \end{vmatrix} - k \begin{vmatrix} \phi_2 & \phi_4 \\ \phi_6 & \phi_8 \end{vmatrix} - l \begin{vmatrix} \phi_3 & \phi_4 \\ \phi_7 & \phi_8 \end{vmatrix} - m \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_5 & \phi_6 \end{vmatrix}$$
$$- n \begin{vmatrix} \phi_1 & \phi_3 \\ \phi_5 & \phi_7 \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_4 \\ \phi_5 & \phi_8 \end{vmatrix} = 0. \tag{9}$$

If we demand that this quadric shall be a sphere, certain well-known relations must exist among the coefficients of the terms of the second degree; by means of these relations we can express the conditions to be satisfied by the defining constants of the linear functions (8) in order that the transformations determined by the two bilinear equations (7) may be line-sphere transformations. It is clear that if every transformation of the family defined by the equations (7) is to transform every straight line into a sphere, then the above-named equations of condition must not contain the parameters k, l, m, n.

The form of the equation (9) shows that the quadric reduces to a sphere without k, l, m, n entering the equations of condition in the following cases and in no other:

1° when any determinant of the matrix

$$\left\| \begin{matrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{matrix} \right\|, \tag{10}$$

where
$$\psi_i \equiv a_i X + b_i Y + c_i Z \equiv \phi_i - d_i \tag{11}$$
reduces to the form
$$\text{const.}\,(X^2 + Y^2 + Z^2), \tag{12}$$

and at the same time the functions ϕ_i corresponding to the ψ_i remaining in the matrix reduce to constants; for example, ϕ_2, ϕ_3, ϕ_6, ϕ_7 constants and the determinant $\psi_1\psi_8 - \psi_4\psi_5$ of the form (12);

2° when all six determinants of the matrix (10) are of the form (12).

3. Examining the first case, and with particular reference to the example noted, we find that the two bilinear equations

$$\left.\begin{aligned} (a_1 X + b_1 Y + c_1 Z + d_1)\,x + d_2 y + d_3 z + a_4 X + b_4 Y + c_4 Z + d_4 = 0, \\ (a_5 X + b_5 Y + c_5 Z + d_5)\,x + d_6 y + d_7 z + a_8 X + b_8 Y + c_8 Z + d_8 = 0, \end{aligned}\right\} \tag{13}$$

where the constants are subject to the conditions

$$\left.\begin{aligned} &a_1 a_8 - a_4 a_5 = b_1 b_8 - b_4 b_5 = c_1 c_8 - c_4 c_5, \\ &a_1 b_8 + b_1 a_8 - a_4 b_5 - b_4 a_5 = 0, \\ &b_1 c_8 + c_1 b_8 - b_4 c_5 - c_4 b_5 = 0, \\ &c_1 a_8 + a_1 c_8 - c_4 a_5 - a_4 c_5 = 0, \end{aligned}\right\} \tag{14}$$

determine a family of ∞^{13} transformations which change all straight lines into spheres. The exponent is readily verified since each equation of (13) is linear and homogeneous in ten constants which are connected by five independent equations (14).

Treating the remaining determinants of the matrix and the corresponding functions in the same manner, we obtain six families of ∞^{13} transformations changing straight lines into spheres.

4. By employing the method of Sophus Lie,* we can show that the preceding line-sphere transformations are contact transformations.

The finite equations expressing the coordinates X, Y, Z, P, Q of the surface element corresponding to the surface element (x, y, z, p, q) as functions of the coordinates of the latter, are given by the resolution of a system of linear equations composed of the two following:

---

* Lie-Engel, "Theorie der Transformationsgruppen," vol. 2.

$$\left.\begin{aligned}\Omega_1 \equiv (a_1 x + a_4) X + (b_1 x + b_4) Y + (c_1 x + c_4) Z \\ + d_1 x + d_2 y + d_3 z + d_4 = 0, \\ \Omega_2 \equiv (a_5 x + a_8) X + (b_5 x + b_8) Y + (c_5 x + c_8) Z \\ + d_5 x + d_6 y + d_7 z + d_8 = 0,\end{aligned}\right\} \quad (15)$$

and the three obtained by eliminating the ratio $\lambda_1 : \lambda_2$ from the system

$$\left.\begin{aligned}\left(\lambda_1 \frac{\partial \Omega_1}{\partial z} + \lambda_2 \frac{\partial \Omega_2}{\partial z}\right) p + \lambda_1 \frac{\partial \Omega_1}{\partial x} + \lambda_2 \frac{\partial \Omega_2}{\partial x} = 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial z} + \lambda_2 \frac{\partial \Omega_2}{\partial z}\right) q + \lambda_1 \frac{\partial \Omega_1}{\partial y} + \lambda_2 \frac{\partial \Omega_2}{\partial y} = 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial Z} + \lambda_2 \frac{\partial \Omega_2}{\partial Z}\right) P + \lambda_1 \frac{\partial \Omega_1}{\partial X} + \lambda_2 \frac{\partial \Omega_2}{\partial X} = 0, \\ \left(\lambda_1 \frac{\partial \Omega_1}{\partial Z} + \lambda_2 \frac{\partial \Omega_2}{\partial Z}\right) Q + \lambda_1 \frac{\partial \Omega_1}{\partial Y} + \lambda_2 \frac{\partial \Omega_2}{\partial Y} = 0;\end{aligned}\right\} \quad (16)$$

in fact X, Y, Z are found by solving the system

$$\Omega_1 = 0; \quad \Omega_2 = 0; \quad AX + BY + CZ + D = 0, \quad (17)$$

where

$$\left.\begin{aligned}A = a_5 e - a_1 f, \quad B = b_5 e - b_1 f, \quad C = c_5 e - c_1 f, \\ D = d_5 e - d_1 f - (d_3 d_6 - d_2 d_7) p, \quad e = d_2 + d_3 q, \quad f = d_6 + d_7 q;\end{aligned}\right\} \quad (18)$$

and P, Q have the values

$$\left.\begin{aligned}P &= \frac{1}{R} \{(a_1 x + a_4)(d_6 + d_7 q) - (a_5 x + a_8)(d_2 + d_3 q)\}, \\ Q &= \frac{1}{R} \{(b_1 x + b_4)(d_6 + d_7 q) - (b_5 x + b_8)(d_2 + d_3 q)\}, \\ R &= (c_5 x + c_8)(d_2 + d_3 q) - (c_1 x + c_4)(d_6 + d_7 q).\end{aligned}\right\} \quad (19)$$

5. Relative to the transformations defined by the equations (13) and (14), it

is interesting to observe further that the absolute term of the equation (9) becomes zero independently of k, l, m, n if the equations

$$\left.\begin{array}{lll} d_1 d_6 - d_2 d_5 = 0, & d_1 d_7 - d_3 d_5 = 0, & d_1 d_8 - d_4 d_5 = 0, \\ d_2 d_7 - d_3 d_6 = 0, & d_2 d_8 - d_4 d_6 = 0, & d_3 d_8 - d_4 d_7 = 0, \end{array}\right\} \tag{20}$$

hold; then the transformations determined by (13) and (14) change straight lines into points if the additional conditions

$$d_1 : d_2 : d_3 : d_4 = d_5 : d_6 : d_7 : d_8 \tag{21}$$

are satisfied.

The following particular case is especially noteworthy. Assigning the values

$$\left.\begin{array}{l} a_1 = a_8 = b_1 = b_8 = c_4 = c_5 = d_1 = d_2 = d_4 = d \; = d_7 = d_8 = 0, \\ a_4 = a_5 = c_1 = -c_8 = d_3 = d_6 = 1, \quad b_4 = -b_5 = i = \sqrt{-1}, \end{array}\right\} \tag{22}$$

to the constants in the equations (13) or (15) and remarking that this system of constants (22) satisfies the equations of condition (14), we have the celebrated correspondence of Lie determined by the *æquationes directrices*

$$Zx + z + X + iY = 0, \quad (X - iY)x + y - Z = 0, \tag{23}$$

and studied in his well-known memoir,* "Ueber Complexe, insbesondere Linien- und Kugel-Complexe."

6. Returning now to the second case in which the quadric (9) can become a sphere for all values of k, l, m, n, namely, when all six determinants of the matrix (10) reduce to the form (12), we find that, although its equations of condition are apparently more numerous, yet it yields a more remarkable and extensive family of line-sphere contact-transformations.

In fact, the equations

$$\left.\begin{array}{l} a_\iota a_\sigma - a_\rho a_j = b_\iota b_\sigma - b_\rho b_j = c_\iota c_\sigma - c_\rho c_j, \\ a_\iota b_\sigma + b_\iota a_\sigma - a_\rho b_j - b_\rho a_j = 0, \\ b_\iota c_\sigma + c_\iota b_\sigma - b_\rho c_j - c_\rho b_j = 0, \\ c_\iota a_\sigma + a_\iota c_\sigma - c_\rho a_j - a_\rho c_j = 0, \end{array}\right\} \tag{24}$$

---

* Mathematische Annalen, vol. 5.

which are necessary and sufficient that the determinant

$$\psi_i \psi_\sigma - \psi_\rho \psi_j \tag{25}$$

shall be changed to the form (12), possess the symmetrical solution

$$\left.\begin{array}{llll} b_i = a_i\sqrt{-1}, & b_\rho = -a_\rho\sqrt{-1}, & b_j = a_j\sqrt{-1}, & b_\sigma = -a_\sigma\sqrt{-1}, \\ c_i = a_\rho, & c_j = a_\sigma, & c_\rho = -a_i, & c_\sigma = -a_j; \end{array}\right\} \tag{26}$$

hence, the functions ψ have the forms

$$\begin{array}{l} \psi_j \quad \equiv a_j X + i a_j Y + c_j Z, \\ \psi_{j+4} \equiv c_j X - i c_j Y - a_j Z, \end{array} \quad j = 1, 2, 3, 4, \; i = \sqrt{-1}, \tag{27}$$

and, therefore, the two bilinear equations

$$x\omega_1 + y\omega_2 + z\omega_3 + \omega_4 = 0, \quad x\omega_5 + y\omega_6 + z\omega_7 + \omega_8 = 0, \tag{28}$$

where
$$\omega_i = \psi_i + d_i, \quad i = 1, 2, \ldots, 8 \tag{29}$$

determine a family of ∞^{15} transformations which change straight lines into spheres.

That these ∞^{15} transformations are contact-transformations can be established in the following manner: Combining Lie's transformation (23) expressed by the two bilinear equations

$$Zx_1 + z_1 + X + iY = 0, \quad (X - iY)x_1 + y_1 - Z = 0, \tag{30}$$

with all the transformations of the general projective group

$$\left.\begin{array}{l} \rho x_1 = \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1, \\ \rho y_1 = \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2, \\ \rho z_1 = \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3, \\ \rho \quad = \alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4, \end{array}\right\} \tag{31}$$

the resulting ∞^{15} transformations form a family of line-sphere contact-transformations determined by two bilinear equations

$$\left.\begin{array}{l} (\alpha_4 X + i\alpha_4 Y + \alpha_1 Z + \alpha_3)x + (\beta_4 X + i\beta_4 Y + \beta_1 Z + \beta_3)y + \ldots . = 0, \\ (\alpha_1 X - i\alpha_1 Y + \alpha_4 Z + \alpha_2)x + (\beta_1 X - i\beta_1 Y - \beta_4 Z + \beta_2)y + \ldots . = 0, \end{array}\right\} \tag{31}$$

which equations are precisely of the form (28).

Hence, the equations (28) define a fifteen-parameter family of line-sphere contact-transformations (Σ) of such a nature that they are derived from the transformation (23) (Λ) of Lie and the transformations (Π) of the general projective group (30) by means of the symbolic equation

$$\Sigma = \Pi\Lambda. \tag{32}$$

7. This note limits itself to the determination of those line-sphere transformations which are defined by two bilinear equations between the point coordinates of the corresponding spaces, both spaces being ordinary; but it may be permitted to add several postscripts calling attention to other generalizations of Lie's transformation.

1°. A six-parameter family of line-sphere contact-transformations is obtained by generalizing the form*

$$\left.\begin{aligned} a &= x + yi, & b &= z + R, \\ q &= x - yi, & p &= R - z, \end{aligned}\right\} \tag{33}$$

given to Lie's transformation by Professor Darboux.

The equations

$$a_i = \varkappa_i \alpha + \lambda_i \beta + \mu_i \gamma + \nu_i \rho, \tag{34}$$

establish a correspondence between the straight lines (a_1, a_2, a_3, a_4) and the spheres (α, β, γ, ρ) in such a manner that intersecting straight lines are transformed into tangent spheres if the ten equations

$$\left.\begin{aligned} \varkappa_1\varkappa_2 - \varkappa_3\varkappa_4 = \ldots\ldots = \nu_1\nu_2 - \nu_3\nu_4 = 1, \\ \varkappa_1\lambda_2 + \lambda_1\varkappa_2 - \varkappa_3\lambda_4 - \lambda_3\varkappa_4 = \ldots\ldots = 0, \end{aligned}\right\} \tag{35}$$

are satisfied, since these equations express the necessary and sufficient conditions that the quadratic form

$$(a_1 - a_1')(a_4 - a_4') - (a_2 - a_2')(a_3 - a_3') \tag{36}$$

shall be changed by the transformation (34) into the quadratic form

$$(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2 - (\rho - \rho')^2. \tag{37}$$

2°. The method of the note may be employed to construct certain anomalous correspondences of a similar form in spaces of higher dimensions; these cease to be of any but formal interest in other than ordinary spaces, since for a space the number of whose dimensions is greater than three, the number of right lines is infinitely greater than the number of spheres; an example of such a correspondence in a four-dimensional space is the transformation defined by the two following bilinear equations:

---

* Darboux, "Théorie des Surfaces," vol. 1, §157.

$$\left.\begin{aligned}(Z-iT)\,x+z+(X+iY)(t+1)&=0,\\(X-iY)\,x+y-(Z+iT)(t+1)&=0;\end{aligned}\right\}\qquad(38)$$

it changes straight lines into spheres; it is an incomplete contact-transformation which degenerates into Lie's transformation (23) on making the fourth dimension zero.

3°. By employing the results of the note, an infinite number of infinite families of line-sphere contact-transformations of ordinary space can be constructed; these are no longer defined by two bilinear *æquationes directrices.*

PRINCETON, NEW JERSEY, *November* 1, 1899.

The Ellipsograph of Proclus.[*]

BY E. M. BLAKE.

If a plane σ containing two points E and E_1 moves upon a coincident plane σ_1 containing two straight lines g and g_1 so that E remains upon g and E_1 upon g_1, the two planes form a mechanism possessing the following well-known properties: Every point of σ traces an ellipse upon σ_1, and every point of σ_1 traces a limaçon upon σ.[†] A circle c of radius a in σ rolls upon the inner side of a circle c_1 of radius $2a$ in σ_1. Every point of c describes a straight line passing through the center of c_1. Any two of these lines, with the points which generate them, can be taken for g, g_1 and E, E_1 in defining the movement.

The object of the present paper is to make a brief study of—1° the curves generated by the points of σ and σ_1; 2° the ruled surfaces generated by any straight line carried by σ or σ_1 and not parallel to them; 3° the curves enveloped by any straight line of σ or σ_1; 4° the developables enveloped by carried planes.

The point loci have been given by Cayley[‡] and Schell.[§] The line loci for the ellipsograph have been determined by Burmester,[||] although he omits their equation, but those for the inverse movement are believed to be given here for the first time. The envelope of the straight line of σ which passes through E and E_1 when g and g_1 are at right angles, was found by Chasles to be the four-cusped hypocycloid.* The only other lines of σ, the equations of whose envelopes have been previously determined, are those forming a square with EE_1.†

---

[*] Read before the Chicago Section of the American Mathematical Society, April 9, 1898, at which eight thread models of surfaces described in the paper were exhibited. This embodies the author's paper, "Upon a Ruled Surface of the Fourth Order Mechanically Generated," read before the Society December 31, 1897. Paper revised, Ithaca, January, 1900.

[†] The discovery of the first property is accredited to Proclus; Chasles, "Aperçu historique," 2d ed., p. 49. The well-known chuck for turning figures with elliptical cross-sections, invented by Leonardo da Vinci, is an application of the mechanism. For other historical notes, see Burmester. "Lehrbuch der Kinematik I," Leipzig, 1888, pp. 36–42; A. v. Braunmühl, "Studie über Curvenerzeugung" in the "Katalog mathematischer Modelle," by Dyck.

[‡] "On the Kinematics of a Plane." Quarterly Journal, XVI, 1878, pp. 1–8.

[§] "Theorie der Bewegung und Kräfte I, pp. 227–230.

[||] "Kinematische Flächen erzeugung vermittelst cylindrischer Rollung." Zeit. für Math. u. Phys. XXXIII, 1888, pp. 337–348. The surface with real and distinct nodal straight lines is also given by Mannheim, Comptes Rendus de l'Acad. de Paris, LXXVI, 1873, pp. 635–639; Bulletin de la Soc. Math. de France, I, 1875, pp. 106–114. Menzel, "Ueber die Bewegung einer starren Geraden." Dissertation, Münster, 1891.

Point and Straight Line Loci of the Ellipsograph.

Denoting by (x, y) the rectangular coordinates of any point of σ, taking for origin the center O of c, and by (x_1, y_1) the same for any point of σ_1 taking for origin the center O_1 of c_1; the equation

$$x_1^2(x^2 - 2ax + a^2 + y^2) - 4ax_1y_1y + y_1^2(x^2 + 2ax + a^2 + y^2) = (x^2 + y^2 - a^2)^2 \quad (1)$$

represents the ellipse in σ_1 traced by the point (x, y) of σ or the limaçon traced by (x_1, y_1) upon σ.‡

From the equation we infer the following properties of the congruence of ellipses described by the points of σ: They all have the point O_1 for center, and rotation through half the angle whose tangent is $\frac{y}{x}$ brings their axes into coincidence with the coordinate axes. The points of a circumference whose center is O describe congruent ellipses, which degenerate to the diameters of c_1 when the circumference is c. The point O describes a circle of radius a. The points of a straight line through O describe ellipses whose axes lie upon two fixed lines.§

To obtain the equation of the surfaces generated by the straight lines carried by σ and oblique to it, we proceed as in a former paper.‖ Take axes of z and z_1 passing respectively through O and O_1 and perpendicular to σ and σ_1. Let l be a representative generator parallel to yOz, passing through the point

---

* "Aperçu historique," p. 69. The author also remarks that the envelope of any other straight line is the involute of a hypocycloid.

† J. B. Pomey, "Enveloppes des côtés d'un carré invariable dont deux sommets décrivent deux droites rectangulaires." Nouvelles Annales (3), V, 1886, pp. 520–530.

‡ Cayley and Schell, loc. cit.

§ For greater detail see J. S. et N. Vaněček. "Sur les ellipses décrites par les points invariablements liés à un segment constant et sur une surface circulaire du huitième ordre." Bulletin de la Soc. Math. de France, XI, 1883, pp. 76–88.

‖ American Journal, XXI, 1899, pp. 260–261.

$(p, 0)$ of σ and making with that plane the angle whose cotangent is s; its locus is

$$x_1^2(p^2 - 2ap + a^2 + s^2 z_1^2) - 4as\,x_1 y_1 z_1 + y_1^2(p^2 + 2ap + a^2 + s^2 z_1^2) = (p^2 + s^2 z_1^2 - a^2)^2 \quad (2)$$

obtained from (1) by substituting for x and y respectively p and sz_1.

There is no difficulty now in establishing the following theorem: *Any straight line oblique to σ and carried by it generates a quartic scroll having a real isolated double generator in the plane σ_1 at infinity and two other nodal straight lines which are: either real and distinct, coincident, or imaginary according as l' (the orthogonal projection of l upon σ) intersects c in two real, two coincident, or two imaginary points.*

Point and Straight Line Loci of the Inverse of the Ellipsograph.

Regarding the plane σ as fixed, the points (x_1, y_1) of σ_1 describe upon it a congruence of limaçons, defined by equation (1) and possessing the following properties: Any point on c_1 describes a cardioide, any point within a limaçon having a node, and any point without one having a conjugate point. All points of a circumference with center at O_1 trace congruent curves. They may be brought into coincidence by rotation about O through twice the angle whose tangent is $\frac{y_1}{x_1}$. The point O_1 describes the circle c twice during a cycle of the movement. The real double points of all the curves are upon c. The points of a line through O_1 making the angle θ with the axis of x_1 generate curves with the same line of symmetry making the angle 2θ with Ox.

As in the preceding case, we have for the locus of l the equation

$$p^2(x^2 - 2ax + a^2 + y^2) - 4apsyz + s^2 z^2 (x^2 + 2ax + a^2 + y^2) = (x^2 + y^2 - a^2)^2, \quad (3)$$

and the following theorem: *Any straight line oblique to σ_1 and carried by it generates a quartic scroll having: if l' passes through O_1, a nodal circle in σ (or a parallel plane) and an intersecting nodal straight line; and if l' does not pass through, a nodal cubical ellipse. Upon the latter are two imaginary pinch-points, and in addition either two real, two coincident, or two imaginary pinch-points according as l' intersects c_1 in two real, two coincident, or two imaginary points.*

Singularities of Point Loci.

The circle of inflexions* for the ellipsograph is c, and hence, any point upon it is always describing an inflexion upon its trajectory, i. e., the trajectory is a straight line. The circle of inflexions for the inverse movement has the same radius as c, and passes, during a complete cycle of the movement, once around c_1, remaining tangent to it and to the concentric circle of radius $4a$. Hence, the limaçon generated by a point of σ_1 between these two circumferences has two real points of inflexion. The two inflexions merge into one of the next higher order if the tracing-point be taken upon the circumference of radius $4a$.

The movement of a plane σ upon another σ_1 is either a rotation about a fixed point, a movement such that all straight lines of σ remain continually parallel to their initial positions, or the result of rolling a curve C of σ upon a curve G_1 of σ_1. Movements of the latter type are the only ones which give rise to any but trivial problems in loci and envelopes. It is easy to show that *the movement defined by the ellipsograph is the only movement of the last type which causes all points of σ to describe conics.* For any point of C describes a curve with at least one cusp. A conic having a cusp consists of two coincident straight lines. Hence, at least two points of σ describe straight lines, which defines the movement of the ellipsograph.

The Curves Enveloped by Straight Lines carried Parallel to the Plane of Movement, and the Developables Enveloped by carried Planes.

We will commence by finding the envelope of the line $y = d$ carried by σ. Assume for convenience $a = \frac{1}{2}$, then the equation of one of the positions of $y = d$ in σ_1 is

$$\frac{x_1}{b'} + \frac{y_1}{b} = 1 + \frac{d}{bb'}\,; \quad b^2 + b'^2 = 1,$$

b and b' being respectively the intercepts of $y = 0$ upon the axes of x_1 and y_1. Eliminating b', we have

$$b^4 - 2y_1\, b^3 + (x_1^2 + y_1^2 - 1)\, b^2 + 2\,(y_1 - dx_1)\, b + d^2 - y_1^2 = 0.$$

---

[*] Koenigs, "Leçons de cinématique," I, pp. 144–154; Schoenflies, "La géométrie du mouvement." Paris, 1893, Chapter I.

The discriminant of this with respect to b equated to zero gives

$$(x_1^2+y_1^2)^3-(3+d^2)y_1^4-18dx_1y_1^3+(21-2d^2)x_1^2y_1^2-18dx_1^3y_1-(3+d^2)x_1^4$$
$$+(3+20d^2)y_1^2-(36d-16d^3)x_1y_1+(3+20d^2)x_1^2-16d^4+8d^2-1=0, \quad (4)$$

the envelope required, and $(x_1y_1-d)^2=0$. The latter may be regarded as the locus of nodes, which should occur twice. It is due to the enveloping line having two positions for any value of b.

The equation (4) represents, except for a rotation about the origin, the curve enveloped by any straight line of σ. It also represents with the same reservation the cylinder enveloped by any plane carried by, and perpendicular to, σ. The surface enveloped by any plane making an acute angle with σ is evidently congruent to one enveloped by a plane passing through $y=0$ and making the same angle with σ. They may be brought into coincidence by rotation about and translation along the axis of z_1. We proceed immediately to the determination and discussion of the equation of the latter surface, in the course of which we shall be in a position to determine the character of its plane sections parallel to σ_1, which are the same as the curves (4).

The equation of the surface enveloped by the plane which passes through $y=0$ and makes with σ the angle whose cotangent is s, is obtained from (4) by substituting sz_1 for d. The result is

$$(x_1^2+y_1^2)^3-(3+s^2z_1^2)y_1^4-18sx_1y_1^3z_1+(21-2s^2z_1^2)x_1^2y_1^2$$
$$-18sx_1^3y_1z_1-(3+s^2z_1^2)x_1^4+(3+20s^2z_1^2)y_1^2-(36sz_1-16s^3z_1^3)x_1y_1$$
$$+(3+20s^2z_1^2)x_1^2-16s^4z_1^4+8s^2z_1^2-1=0. \quad (5)$$

Rotating the coordinate axes through 45° about the axis of z_1, gives this equation the more convenient form

$$(x_1^2+y_1^2)^3-(3+s^2z_1^2)(x_1^2+y_1^2)^2+\tfrac{27}{4}(x_1^2-y_1^2)^2-9sz_1(x_1^4-y_1^4)$$
$$+(3+20s^2z^2)(x_1^2+y_1^2)-(18sz_1-8s^3z_1^3)(x_1^2-y_1^2)-16s^4z_1^4$$
$$+8s^2z_1^2-1=0, \quad (6)$$

to which the discussion which follows will apply.

The surface is of the sixth order and symmetrical with respect to $x_1=0$ and $y_1=0$. Its intersection with $y_1=0$ consists of the two straight lines $z_1=\pm\frac{x_1}{s}-\frac{1}{2s}$ and the parabola $x_1^2-4sz_1+2=0$ taken twice. The lines are tangent to the parabola at $\left(\pm 2, \frac{3}{2s}\right)$. The section by $x_1=0$ consists of the lines $z_1 = \pm \frac{y_1}{s} + \frac{1}{2s}$ and the parabola $y_1^2 + 4z_1 + 2 = 0$ taken twice. The lines are tangent to the parabola at $\left(\pm 2, -\frac{3}{2s}\right)$. The two parabolas are nodal lines of the surface. Each has a segment lying upon the surface and an isolated segment. The former contains the vertex of the parabola, and is included between its points of tangency with the straight lines mentioned above.

The edge of regression is readily verified to be the intersection of the two cylinders

$$\left.\begin{aligned} x_1^2 &= \tfrac{4}{27}\left(sz_1 + \tfrac{3}{2}\right)^3, \\ y_1^2 &= -\tfrac{4}{27}\left(sz_1 - \tfrac{3}{2}\right)^3, \end{aligned}\right\} \tag{7}$$

whose right sections are semi-cubic parabolas. Its projecting cylinder upon α_1 is

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}. \tag{8}$$

By equations (7) the curve is upon the hyperboloid of revolution*

$$x_1^2 + y_1^2 - \tfrac{4}{3} s^2 z_1^2 = 1.$$

Any two successive positions of a plane β, subject to a plane movement, intersect in a straight line an element of its envelope making the same (constant) angle with the plane of movement (σ and σ_1) as β does. Hence, the tangents of the edge of regression of the envelope of β make a constant angle with the plane of movement, i. e., the edge is a curve of constant slope.† The slope of the curve (7) is the reciprocal of s.

The edge of regression has cusps at the four points $\left(2, 0, \frac{3}{2s}\right)$, $\left(0, 2, -\frac{3}{2s}\right)$, $\left(-2, 0, \frac{3}{2s}\right)$, $\left(0, -2, -1, \frac{3}{2s}\right)$, given in order along it. The portions between are convex toward the origin. The projecting cylinders (7) are each tangent to one of the nodal parabolas. Hence, at each cusp of the edge of regression there is a tangent nodal parabola.

Due to the convexity toward the origin of the edge of regression, it is possible to pass a plane between the origin on one side and three of its cusps on the other, so near the latter that the edge of regression pierces the plane in six real points, two in the vicinity of each cusp. The edge of regression is thus of the sixth order, and each piercing point is a cusp on the curve of intersection of the plane and developable. In addition to these cusps, the plane section has four double points, the piercing points of the two nodal parabolas. For the order six, with four double points and six cusps, Plücker's equations give: class four, no inflexions and three double tangents. The multiple points accounted for must be all, since an additional node or cusp would reduce the class to two or one. The two parabolas are the only nodal lines. The value four for the class of the surface, that being the same as the class of its plane sections, is verified by a theorem of Darboux* which states the class of a surface enveloped by a plane having one degree of freedom to be equal to the order of the trajectory of a point subjected to the inverse movement. The inverse gives in this case the limaçon.

---

* For this result I am indebted to the "Referee," who has shown that the edge of regression of the envelope of a plane, when the centrodes of the movement are circles, is upon a quadric of revolution.

† The "Referee."

We are now prepared to study the sections of the surface (6) by planes parallel to σ_1. As remarked, they are the same as the envelopes of the straight lines of σ which are parallel to $y = 0$. The section by $z_1 = 0$ is a four-cusped hypocycloid, the cuspidal tangents bisecting the angles between the coordinate axes. Its cusps are at the middle points of the arcs of the hypocycloid

$$x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

of double its size, which is the projection of the edge of regression. Since $z_1 = 0$ intersects each of the nodal parabolas in two imaginary points, the section has four imaginary double-points. Besides the four real cusps, it has two which are imaginary, the remaining points of intersection of the plane with the edge of regression. As the value of c increases from zero, the four real cusps of the section by $z_1 = c$ move symmetrically away from the axis of y_1 going along the curve $x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = 2^{\frac{2}{3}}$ towards its cusps $(\pm 2, 0)$. The curve of section has no real nodes until it becomes tangent to the nodal parabola $x_1^2 - 4sz_1 + 2 = 0$, $y_1 = 0$ for $c = \dfrac{1}{2s}$ when the section has a tac-node at the origin. By further increasing c, the tac-node resolves itself into two real nodes which move along the axis of x_1 away from the origin in opposite directions. For $c = \dfrac{3}{2s}$, two cusps and a node unite at each of the points $(\pm 2, 0)$

---

* Koenigs, "Leçons de cinématique, p. 358.

and the curve which has no visible singularities resembles an ellipse whose major axis is on the axis of x_1. The singular points of this curve are the cusps of the edge of regression. For values of c greater than $\frac{3}{2s}$ the sections resemble ellipses, their only real singularities being two conjugate points within and on the axis of x_1. The curves become more and more nearly circular as c increases in value. The curve of section by $z_1 = -c$, when rotated in its plane through a right angle, is the same as the section by $z_1 = c$.

In the case of any plane movement whose centrodes are circles, the characteristic of the envelope of a carried plane through a diameter AB of the moving centrode, is a straight line passing through the center of the centrode and perpendicular to AB. Hence, the projection of the edge of regression upon the plane of movement is the evolute of the cycloid enveloped by AB, i. e., a similar cycloid; and all plane sections of the envelope parallel to the plane of movement are involutes of this projection. For the special movement under consideration, AB envelopes

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

and its evolute is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{2}{3}},$$

which agrees with equation (8). The involutes of this are the sections of (6) by $z_1 =$ const.[*]

Turning, in conclusion, to the inverse of the ellipsograph, a straight line of σ_1 envelopes a circle in σ whose center is upon c, for it is parallel to a diameter of c_1, and every diameter of c_1 passes through a fixed point of c. The envelope of a carried plane is a cone or a cylinder of revolution.

---

[*] My attention was called to this by the "Referee."

Displacements Depending on One, Two, ..., k Parameters in a Space of n Dimensions.

BY N. J. HATZIDAKIS.

1. Professor Craig has recently considered (this Journal, vol. XX, No. 2, April, 1898) the displacements in a space of four dimensions and generalized the theory of M. Darboux. In the present short paper I shall examine the general case of displacements in a space of n dimensions.

2. In the ordinary space of three dimensions, the curves and the surfaces correspond to the displacements depending on one or two parameters. No relations exist between the kinematic elements of the curves; but for the surfaces six relations exist between the ten kinematic elements (the two others, ζ and ζ_1, are $= 0$). These relations are the *fundamental kinematic equations* of the surfaces. This agrees perfectly with the usual analytical theory of the surfaces, which gives *three* fundamental equations between the *six* elements of Gauss, E, F, G, D, D', D''.[*] It suffices to remember the relations existing between the kinematic and the analytical elements.[†]

3. In the general case of a linear space of n dimensions, $n-1$ kinds of displacements are to be considered, viz., those which depend on one, two, ..., $n-1$ parameters. To the *one-parametric* displacements correspond the curves of this space; to the *two-parametric*, the surfaces; to the *three-parametric*, the *hypersurfaces*, or *manifoldnesses* of three dimensions, etc.; to the *k-parametric* displacements correspond the *manifoldnesses* (Variétés, Gebilde) of k dimensions. No

---

[*] Darboux, "Surfaces," vol. III, pp. 247–8, or Crelle's Journal, vol. LXXXVIII, p. 69 (1880).

[†] Darboux, "Surfaces," vol. II, pp. 376 and 379.

relations exist for the kinematic elements of the curves in this general space; for all the other manifoldnesses, however, there exist many relations which we shall now find.

---

4. In the n-dimensional space, let $OX_1 X_2 \dots . X_n$ be a system of fixed axes and $ox_1 x_2 \dots . x_n$ a movable system. Suppose both systems orthogonal.

The n equations

$$x_1 = 0, \quad x_2 = 0, \dots . , x_n = 0,$$

(each separately) give n $(n-1)$-dimensional linear spaces, viz., the $(n-1)$-dimensional coordinate spaces in the moving system. Further, the $\frac{n(n-1)}{2}$ systems of equations

$$\begin{array}{lllr} x_1 = 0, x_2 = 0; & x_1 = 0, x_3 = 0; & \dots . ; & x_1 \quad = 0, x_n = 0; \\ & x_2 = 0, x_3 = 0; & \dots . ; & x_2 \quad = 0, x_n = 0; \\ & & \dots\dots\dots & \dots\dots\dots\dots \\ & & & x_{n-1} = 0, x_n = 0 \end{array}$$

(each separately), give $\frac{n(n-1)}{2}$ $(n-2)$-dimensional coordinate spaces in the moving system. Generally, we find $\frac{n(n-1)\dots . (n-k+1)}{k!}$ coordinate spaces of the $(n-k)^{\text{th}}$ dimension. For $k = n-1$, we have the n axes. But among all these kinds of spaces, those of the $(n-2)^{\text{th}}$ dimension correspond, for the rotations, to the axes of the three-dimensional space. I shall, consequently, call the figure formed by these spaces the *moving polyhedron* (par excellence).

5. Let, now, the following be the *schema* of the cosines of the movable axes with the fixed

$$\begin{array}{c|cccc} & x_1 & x_2 \dots . & x_{n-1} & x_n \\ \hline X_1 & \alpha_1 & \alpha_2 \dots . & \alpha_{n-1} & \alpha_n \\ X_2 & \beta_1 & \beta_2 \dots . & \beta_{n-1} & \beta_n \\ \dots . & \dots & \dots & \dots & \dots \\ X_{n-1} & \mu_1 & \mu_2 \dots . & \mu_{n-1} & \mu_n \\ X_n & \nu_1 & \nu_2 \dots . & \nu_{n-1} & \nu_n \end{array} \qquad (1)$$

The determinant of these cosines being orthogonal, we have the relations

$$\left.\begin{array}{l}
\sum_{i=1}^{n}\alpha_i^2=1,\quad \sum_{i=1}^{n}\beta_i^2=1,\quad \sum_{i=1}^{n}\gamma_i^2=1,\;\ldots\ldots,\; \sum_{i=1}^{n}\nu_i^2=1,\\
\qquad\qquad \sum_{i=1}^{n}\alpha_i\beta_i=0,\quad \sum_{i=1}^{n}\alpha_i\gamma_i=0,\;\ldots\ldots,\; \sum_{i=1}^{n}\alpha_i\nu_i=0,\\
\qquad\qquad\qquad\qquad \sum_{i=1}^{n}\beta_i\gamma_i=0,\;\ldots\ldots,\; \sum_{i=1}^{n}\beta_i\nu_i=0,\\
\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\\
\qquad\qquad\qquad\qquad\qquad\qquad \sum_{i=1}^{n}\mu_i\nu_i=0,
\end{array}\right\} \qquad (2)$$

or the equivalent set

$$\left.\begin{array}{l}
\sum\alpha_1^2=1,\quad \sum\alpha_2^2=1,\quad \sum\alpha_3^2=1,\;\ldots\ldots,\; \sum\alpha_n^2=1,\\
\qquad\qquad \sum\alpha_1\alpha_2=0,\quad \sum\alpha_1\alpha_3=0,\;\ldots\ldots,\; \sum\alpha_1\alpha_n=0,\\
\qquad\qquad\qquad\qquad \sum\alpha_2\alpha_3=0,\;\ldots\ldots,\; \sum\alpha_2\alpha_n=0,\\
\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\\
\qquad\qquad\qquad\qquad\qquad\qquad \sum\alpha_{n-1}\alpha_n=0,
\end{array}\right\} \qquad (2')$$

where the Σ's are extended over all the cosines having the same *index*.

6. In the general motion of a system consisting of a rotation and a translation, we consider at first the case of the rotation, supposing that the moving axes have a fixed origin, namely, that of the $OX_1X_2\ldots\ldots X_n$. We have then

$$V_{x_i}=\alpha_iV_{X_1}+\beta_iV_{X_2}+\ldots\ldots+\nu_iV_{X_n}+\frac{dx_i}{dt},$$
$$(i=1,\ 2,\ \ldots\ldots,\ n)$$

$$\left.\begin{array}{ll}
\text{or} & V_{x_i}=\alpha_i\sum_{j=1}^{n}x_j\frac{d\alpha_j}{dt}+\beta_i\sum_{j=1}^{n}x_j\frac{d\beta_j}{dt}+\ldots\ldots+\nu_i\sum_{j=1}^{n}x_j\frac{d\nu_j}{dt}+\frac{dx_i}{dt},\\
\text{viz.,} & V_{x}=x_1\sum\alpha_i\frac{d\alpha_1}{dt}+x_2\sum\alpha_i\frac{d\alpha_2}{dt}+\ldots\ldots+x_n\sum\alpha_i\frac{d\alpha_n}{dt}+\frac{dx_i}{dt},
\end{array}\right\} \qquad (3)$$

the Σ's extended over all the cosines having the same index. V_{x_i} denotes the components of the absolute velocity of a point x_i relative to the moving axes.[*]

If we now write for the sake of brevity

$$\left.\begin{array}{r} \sum \alpha_2 \frac{d\alpha_1}{dt} = p_{12},\ \sum \alpha_3 \frac{d\alpha_1}{dt} = p_{13},\ \ldots,\ \sum \alpha_n \frac{d\alpha_1}{dt} = p_{1n}, \\ \sum \alpha_3 \frac{d\alpha_2}{dt} = p_{23},\ \ldots,\ \sum \alpha_n \frac{d\alpha_2}{dt} = p_{2n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \sum \alpha_n \frac{d\alpha_{n-1}}{dt} = p_{n-1\,n}, \end{array}\right\} \qquad (4)$$

equations (3) take the form

$$\left.\begin{array}{l} V_{x_1} = \frac{dx_1}{dt} \qquad\qquad - x_2 p_{12} - x_3 p_{13} - \ldots\ldots\ldots - x_n p_{1n}, \\ V_{x_2} = \frac{dx_2}{dt} + x_1 p_{12} \qquad\qquad - x_3 p_{23} - \ldots\ldots\ldots - x_n p_{2n}, \\ V_{x_3} = \frac{dx_3}{dt} + x_1 p_{13} + x_2 p_{23} \qquad\qquad - x_4 p_{34} - \ldots\ldots - x_n p_{3n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_{x_{n-1}} = \frac{dx_{n-1}}{dt} + x_1 p_{1\,n-1} + x_2 p_{2\,n-1} + \ldots\ldots + x_{n-2} p_{n-2\,n-1} - x_n p_{n-1\,n}, \\ V_{x_n} = \frac{dx_n}{dt} + x_1 p_{1n} + x_2 p_{2n} + \ldots\ldots\ldots + x_{n-1} p_{n-1\,n}. \end{array}\right\} \qquad (5)$$

7. It is now very easy to find the equations for the cosines. It suffices, of course, to consider the points having coordinates relatively to the axes $OX_1 X_2 \ldots X_n$: 1, 0, ..., 0 ; 0, 1, 0, ..., 0, etc., 0, 0, ..., 0, 1.[†] We find so the following n^2 equations

---

[*] Cf. Appell, "Mécanique," vol. I, pp. 62–3.

[†] Cf. Darboux, "Surfaces," vol. I, p. 4.

$$\left.\begin{aligned}
\frac{d\alpha_1}{dt} &= \sum_{\lambda=2}^{n} \alpha_\lambda p_{1\lambda}, \\
\frac{d\alpha_2}{dt} &= \sum_{\lambda=3}^{n} \alpha_\lambda p_{2\lambda} - \alpha_1 p_{12}, \\
\frac{d\alpha_3}{dt} &= \sum_{\lambda=4}^{n} \alpha_\lambda p_{3\lambda} - \sum_{\tau=1}^{2} \alpha_\tau p_{\tau 3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\alpha_i}{dt} &= \sum_{\lambda=i+1}^{n} \alpha_\lambda p_{i\lambda} - \sum_{\tau=1}^{i-1} \alpha_\tau p_{\tau i}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\alpha_n}{dt} &= \qquad\qquad - \sum_{\tau=1}^{n-1} \alpha_\tau p_{\tau n},
\end{aligned}\right\} \tag{6}$$

and $n - 1$ other sets of equations for the other cosines $\beta, \gamma, \ldots\ldots, \mu, \nu$.

The $\frac{n(n-1)}{2}$ quantities p, introduced above, are again, as in the cases of the three or the four dimensions, the components of rotation about the $\frac{n(n-1)}{2}$ faces of the *moving polyhedron* which we have considered above. It is very easy to show this in a manner quite analogous to that of Mr. Cole* (for the space of four dimensions), and we can again express the cosines of these faces in terms of the cosines of the axes, but I omit writing the demonstration, as it is not necessary in the following lines,

8. Proceeding now in the same manner as M. Darboux and Professor Craig, we find immediately that the system of differential equations of the first order,

$$\frac{dA_i}{dt} = \sum_{\lambda=i+1}^{n} A_\lambda p_{i\lambda} - \sum_{\tau=1}^{i-1} A_\tau p_{\tau i}, \tag{7}$$
$$(i = 1, 2, 3, \ldots\ldots, n)$$

which the n groups of cosines must satisfy, has always one, and *only* one, solution, when the initial values of the cosines are given; if, further, $A_1, A_2, \ldots, A_n$; $A_1', A_2', \ldots, A_n'$ are two systems of solutions of (7), the quantities

---

* This Journal, vol. XII, pp. 191 et seq. (1890).

$$\sum_{i=1}^{n} A_i^2, \ \sum_{i=1}^{n} A_i A_i', \ \sum_{i=1}^{n} A_i'^2$$

will be constant. We can thus repeat all the reasoning of M. Darboux and show that, when the rotations are given functions of the time (t), the motion is entirely determined; only the position of the fixed axes $OX_1 X_2 \ldots X_n$ is arbitrary; and, further, that, if we have $n-1$ integrals $A_1^{(0)}, A_2^{(0)}, \ldots, A_n^{(0)}$; $A_1^{(1)}, A_2^{(1)}, \ldots, A_n^{(1)}$; etc.; $A_1^{(n-2)}, A_2^{(n-2)}, \ldots, A_n^{(n-2)}$, the general integral A_i will be given by the equations

$$\sum_{i=1}^{n} A_i^2 = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(0)} = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(1)} = \text{const.}, \ldots, \ \sum_{i=1}^{n} A_i A_i^{(n-2)} = \text{const.},$$

or by the following equations, which are all linear in A_i (method of M. Cosserat mentioned by Professor Craig),

$$\sum_{i=1}^{n} A_i A_i^{(0)} = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(1)} = \text{const.}, \ldots, \ \sum_{i=1}^{n} A_i A_i^{(n-2)} = \text{const.},$$

$$\begin{vmatrix} A_1 & A_2 & \ldots & A_n \\ A_1^{(0)} & A_2^{(0)} & \ldots & A_n^{(0)} \\ A_1^{(1)} & A_2^{(1)} & \ldots & A_n^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ A_1^{(n-2)} & A_2^{(n-2)} & \ldots & A_n^{(n-2)} \end{vmatrix} = \text{const.}$$

9. Since the A's are connected by the relation

$$\sum_{i=1}^{n} A_i^2 = \text{const., or (dividing by a convenient constant)} \sum_{i=1}^{n} A_i^2 = 1,$$

we can express them in terms of only $n-1$ variables; we put for this purpose, with Professor Craig,

$$A_1 = \frac{2\Lambda_1}{k^2+1}, \quad A_2 = \frac{2\Lambda_2}{k^2+1}, \ldots, A_{n-1} = \frac{2\Lambda_{n-1}}{k^2+1}, \quad A_n = \frac{k^2-1}{k^2+1},$$

$$\left(k^2 = \sum_{i=1}^{n-1} \Lambda_i^2\right),$$

and we find, after some reductions, the following system containing one equation and one variable less than the system (7).

$$\left.\begin{aligned}
\frac{d\Lambda_1}{dt} &= \sum_{\lambda=2}^{n-1} \Lambda_\lambda p_{1\lambda} + \frac{k^2-1}{2} p_{1n} - \Lambda_1 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n}, \\
\frac{d\Lambda_2}{dt} &= \sum_{\lambda=3}^{-1} \Lambda_\lambda p_{2\lambda} + \frac{k^2-1}{2} p_{2n} - \Lambda_2 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \Lambda_1 p_{12}, \\
\frac{d\Lambda_3}{dt} &= \sum_{\lambda=4}^{n-1} \Lambda_\lambda p_{3\lambda} + \frac{k^2-1}{2} p_{3n} - \Lambda_3 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{2} \Lambda_\tau p_{\tau 3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\Lambda_{n-3}}{dt} &= \sum_{\lambda=n-2}^{n-1} \Lambda_\lambda p_{n-3,\lambda} + \frac{k^2-1}{2} p_{n-3n} - \Lambda_{n-3} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-4} \Lambda_\tau p_{\tau n-3}, \\
\frac{d\Lambda_{n-2}}{dt} &= \Lambda_{n-1} p_{n-2,n-1} + \frac{k^2-1}{2} p_{n-2n} - \Lambda_{n-2} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-3} \Lambda_\tau p_{\tau n-2}, \\
\frac{d\Lambda_{n-1}}{dt} &= \frac{k^2-1}{2} p_{n-1n} - \Lambda_{n-1} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-2} \Lambda_\tau p_{\tau n-1}.
\end{aligned}\right\} \quad (8)$$

We can, of course, apply also the substitution of Mr. J. Eiesland,*

$$A_1 = \frac{\Lambda_1}{\sqrt{k^2+1}}, \quad A_2 = \frac{\Lambda_2}{\sqrt{k^2+1}}, \quad \ldots, \quad A_{n-1} = \frac{\Lambda_{n-1}}{\sqrt{k^2+1}}, \quad A_n = \frac{1}{\sqrt{k^2+1}};$$

$$\left(k^2 = \sum_{i=1}^{n-1} \Lambda_i^2\right)$$

we find, then, the following system of $n-1$ differential equations, equivalent, of course, to the system (8), but a little simplified,

$$\left.\begin{aligned}
\frac{d\Lambda_1}{dt} &= \sum_{\lambda=2}^{n-1} \Lambda_\lambda p_{1\lambda} + p_{1n} + \Lambda_1 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n}, \\
\frac{d\Lambda_2}{dt} &= \sum_{\lambda=3}^{n-1} \Lambda_\lambda p_{2\lambda} + p_{2n} + \Lambda_2 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \Lambda_1 p_{12}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\Lambda_{n-2}}{dt} &= \Lambda_{n-1} p_{n-2,n-1} + p_{n-2,n} + \Lambda_{n-2} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-3} \Lambda_\tau p_{\tau n-2}, \\
\frac{d\Lambda_{n-1}}{dt} &= p_{n-1n} + \Lambda_{n-1} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-2} \Lambda_\tau p_{\tau n-1},
\end{aligned}\right\} \quad (8')$$

---

* This Journal, vol. XX, pp. 245 et seq. (1898).

Each of the two systems (8) and (8′) (or every other system similarly found), constitutes the generalization of the equation of Riccati in the case of n-dimensional spaces. It will be easy to study these systems in a manner analogous to that of Mr. Eiesland (*loc. cit.*). Perhaps I shall find occasion to recur to this subject.

10. An application of the formulæ found above is the case of the motion of the *principal* $\frac{n(n-1)}{2}$*-hedron* of a curve in the space of n dimensions. $n-1$ of the p's are, in this case, equal to the $n-1$ curvatures of the curve, all the others being equal to zero. (See on this subject: Ernesto Cesàro, "Geometria Intrinseca," pp. 226 *et seq.*)

11. Suppose, now, that the origin of the moving axes is not fixed; we shall then only introduce the components, relative to the moving axes ($\xi_1, \ldots, \xi_n$), of the velocity of this origin, and we find the following n equations

$$\frac{dX_1^{(0)}}{dt} = \sum_{i=1}^{n} \alpha_i \xi_i, \quad \frac{dX_2^{(0)}}{dt} = \sum_{i=1}^{n} \beta_i \xi_i, \ldots, \quad \frac{dX_n^{(0)}}{dt} = \sum_{i=1}^{n} \nu_i \xi_i, \tag{9}$$

$X_i^{(0)}$ being the coordinates of the moving origin in the system $OX_1X_2\ldots X_n$. The integration of this system is again always reduced to quadratures, since the coefficients $\alpha_i, \beta_i, \ldots, \nu_i$ are supposed known as functions of the time from the system (6) and ξ_i are given functions of the t.

12. Instead of the system (7) between the cosines of the axes and the rotations, we can find another system of $\frac{n(n-1)}{2}$ equations between the direction cosines P of the faces of the moving polyhedron and the rotations. But the integration of this system can be led back to that of (7); and the relations which this second system gives between the P's and the p's, or between the A's

and the p's (for the P's are given expressions of the A's), must be, of course, the same as those of the system (7). So I omit writing them.

---

13. I shall now consider the case of a two-parametric displacement. Denoting by p the rotations depending on u alone, and by p' those depending on v alone, we obviously have the following equations connecting the p's, the p''s and the cosines:

$$\left(\frac{\partial^2 \alpha_i}{\partial u\,\partial v}\equiv\right)\sum_{\lambda=i+1}^{n}\alpha_\lambda\frac{\partial p_{i\lambda}}{\partial v}+\sum_{\lambda=i+1}^{n}p_{i\lambda}\frac{\partial \alpha_\lambda}{\partial v}-\sum_{\tau=1}^{i-1}\alpha_\tau\frac{\partial p_{\tau i}}{\partial v}-\sum_{\tau=1}^{i-1}p_{\tau i}\frac{\partial \alpha_\tau}{\partial v}$$
$$=\sum_{\lambda=i+1}^{n}\alpha_\lambda\frac{\partial p'_{i\lambda}}{\partial u}+\sum_{\lambda=i+1}^{n}p'_{i\lambda}\frac{\partial \alpha_\lambda}{\partial u}-\sum_{\tau=1}^{i-1}\alpha_\tau\frac{\partial p'_{\tau i}}{\partial u}-\sum_{\tau=1}^{i-1}p'_{\tau i}\frac{\partial \alpha_\tau}{\partial u}\left(\equiv\frac{\partial^2 \alpha_i}{\partial v\,\partial u}\right), \quad (10)$$

or, by the equations (6),

$$\left.\begin{aligned}&\sum_{\lambda=i+1}^{n}\alpha_\lambda\left(\frac{\partial p_{i\lambda}}{\partial v}-\frac{\partial p'_{i\lambda}}{\partial u}\right)-\sum_{\tau=1}^{i-1}\alpha_\tau\left(\frac{\partial p_{\tau i}}{\partial v}-\frac{\partial p'_{\tau i}}{\partial u}\right)\\ &\qquad=\sum_{\lambda=i+1}^{n}p'_{i\lambda}\left(\sum_{l=\lambda+1}^{n}\alpha_l\,p_{\lambda l}-\sum_{t=1}^{\lambda-1}\alpha_t p_{t\lambda}\right)-\sum_{\tau=1}^{i-1}p'_{\tau i}\left(\sum_{l=\tau+1}^{n}\alpha_l p_{\tau l}-\sum_{t=1}^{\tau-1}\alpha_t\,p_{t\tau}\right)-\\ &\qquad-\sum_{\lambda=i+1}^{n}p_{i\lambda}\left(\sum_{l=\lambda+1}^{n}\alpha_l p'_{\lambda l}-\sum_{t=1}^{\lambda-1}\alpha_t\,p'_{t\lambda}\right)+\sum_{\tau=1}^{i-1}p_{\tau i}\left(\sum_{l=\tau+1}^{n}\alpha_l p'_{\tau l}-\sum_{t=1}^{\tau-1}\alpha_t p'_{t\tau}\right),\end{aligned}\right\}(10')$$

where $i = 1, 2, 3, \ldots, n$.

As, now, we have similar equations for the $\beta, \gamma, \ldots, \nu$, we can multiply them by $\alpha_1, \beta_1, \ldots, \nu_1$ and add, or by $\alpha_2, \beta_2, \ldots, \nu_2$ and add, etc., by $\alpha_n, \beta_n, \ldots, \nu_n$ and add; we find thus the following $\frac{n(n-1)}{2}$ *fundamental equations of the two-parametric displacements*, connecting the p's and the p''s

$$
\left.\begin{aligned}
&\frac{\partial p_{1i}}{\partial v}-\frac{\partial p'_{1i}}{\partial u}=\sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{1\lambda}-p_{i\lambda}p'_{1\lambda})-\sum_{\tau=1}^{i-1}(p'_{\tau i}p_{1\tau}-p_{\tau i}p'_{1\tau}),\\
&\qquad\qquad\qquad\qquad(i=2,\ 3,\ \ldots\ldots,\ n)\\
&\frac{\partial p_{23}}{\partial v}-\frac{\partial p'_{23}}{\partial u}=\sum_{\lambda=4}^{n}(p'_{3\lambda}p_{2\lambda}-p_{3\lambda}p'_{2\lambda})+p'_{13}p_{12}-p_{13}p'_{12},\\
&\frac{\partial p_{2i}}{\partial v}-\frac{\partial p'_{2i}}{\partial u}=\sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{2\lambda}-p_{i\lambda}p'_{2\lambda})-\sum_{\tau=3}^{i-1}(p'_{\tau i}p_{2\tau}-p_{\tau i}p'_{2\tau})+p'_{1i}p_{12}-p_{1i}p'_{12},\\
&\qquad\qquad\qquad\qquad(i=4,\ 5,\ \ldots\ldots,\ n)\\
&\frac{\partial p_{34}}{\partial v}-\frac{\partial p'_{34}}{\partial u}=\sum_{\lambda=5}^{n}(p'_{4\lambda}p_{3\lambda}-p_{4\lambda}p'_{3\lambda})+\sum_{\tau=1}^{2}(p'_{\tau 4}p_{\tau 3}-p_{\tau 4}p'_{\tau 3}),\\
&\frac{\partial p_{3i}}{\partial v}-\frac{\partial p'_{3i}}{\partial u}=\sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{3\lambda}-p_{i\lambda}p'_{3\lambda})-\sum_{\tau=4}^{i-1}(p'_{\tau i}p_{3\tau}-p_{\tau i}p'_{3\tau})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sum_{\tau=1}^{2}(p'_{\tau i}p_{\tau 3}-p_{\tau i}p'_{\tau 3}),\\
&\qquad\qquad\qquad\qquad(i=5,\ 6,\ \ldots\ldots,\ n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial p_{k\,k+1}}{\partial v}-\frac{\partial p'_{k\,k+1}}{\partial u}=\sum_{\lambda=k+2}^{n}(p'_{k+1\lambda}p_{k\lambda}-p_{k+1\lambda}p'_{k\lambda})+\sum_{\tau=1}^{k-1}(p'_{\tau k+1}p_{\tau k}-p_{\tau k+1}p'_{\tau k}),\\
&\frac{\partial p_{ki}}{\partial v}-\frac{\partial p'_{ki}}{\partial u}=\sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{k\lambda}-p_{i\lambda}p'_{k\lambda})-\sum_{\tau=k+1}^{i-1}(p'_{\tau i}p_{k\tau}-p_{\tau i}p'_{k\tau})\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sum_{\tau=1}^{k-1}(p'_{\tau i}p_{\tau k}-p_{\tau i}p'_{\tau k}),\\
&\qquad\qquad\qquad\qquad(i=k+2,\ k+3,\ \ldots\ldots,\ n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\partial p_{n-2,\,n-1}}{\partial v}-\frac{\partial p'_{n-2,\,n-1}}{\partial u}=p'_{n-1,\,n}p_{n-2,\,n}-p_{n-1,\,n}p'_{n-2,\,n}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sum_{\tau=1}^{n-3}(p'_{\tau,\,n-1}p_{\tau,\,n-2}-p_{\tau,\,n-1}p'_{\tau,\,n-2}),\\
&\frac{\partial p_{n-2,\,n}}{\partial v}-\frac{\partial p'_{n-2,\,n}}{\partial u}=-p'_{n-1,\,n}p_{n-2,\,n-1}+p_{n-1,\,n}p'_{n-2,\,n-1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sum_{\tau=1}^{n-3}(p'_{\tau n}p_{\tau,\,n-2}-p_{\tau n}p'_{\tau,\,n-2}),\\
&\frac{\partial p_{n-1,\,n}}{\partial v}-\frac{\partial p'_{n-1,\,n}}{\partial u}=\sum_{\tau=1}^{n-2}(p'_{\tau n}p_{\tau n-1}-p_{\tau n}p'_{\tau n-1}).
\end{aligned}\right\}\quad(11)
$$

We have found these equations, equating the coefficients of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ on the left and the right hand. The first equation is given by α_1, $i=2, 3, \ldots, n$ ($i=1$ gives zero), the second by α_2, $i=3$ ($i=1$ gives again an equation of the first line, that with $\alpha=\alpha_1$, $i=2$; $i=2$ gives zero), the third equation is given by α_3, $i=4, 5, \ldots, n$; etc., the equation of the order $2k-2$ is given by α_k, $i=k+1$ ($i=k$ gives zero; $i=1, 2, \ldots, k-1$ gives equations already found), the equation of the order $2k-1$ is given by α_k, $i=k+2, \ldots n$, etc. We have also, finally, an equation by α_{n-1}, $i=n$. Thus, together $\frac{n(n-1)}{2}$ fundamental equations.

The case $n=3$ gives the three equations of M. Darboux, and that of $n=4$ the six equations of Professor Craig.*

14. We can introduce the P's we have mentioned above, but the equations found between the P's, the p's and the p''s are consequences of (10′), and, consequently, we do not find new relations between the p's and p''s.

15. Conversely, whenever the p's and p''s satisfy the system (11), there exists one, and only one, motion having these quantities as rotations. The reasoning is entirely analogous to that of M. Darboux.

16. The integration of the two systems of differential equations for the cosines formed like (6) will be, of course, led back, in this case, to the integration of a simultaneous system of two sets of equations formed like (8) or (8′) and containing two equations and two variables less.

17. Let, now, the moving system have no fixed point; we find in this case

$$\frac{\partial X_1^{(0)}}{\partial u} = \sum_{i=1}^{n} \alpha_i \xi_i, \quad \frac{\partial X_2^{(0)}}{\partial u} = \sum_{i=1}^{n} \beta_i \xi_i, \quad \ldots, \quad \frac{\partial X_n^{(0)}}{\partial u} = \sum_{i=1}^{n} \nu_i \xi_i,$$

---

*Remark that I have put p_{12} equal to the $-p_{12}$ of M. Darboux and Professor Craig, and also $p'_{12} = -p'_{12}$.

and
$$\frac{\partial X_1^{(0)}}{\partial v}=\sum_{i=1}^{n}\alpha_i\xi_i', \quad \frac{\partial X_2^{(0)}}{\partial v}=\sum_{i=1}^{n}\beta_i\xi_i', \ldots\ldots, \quad \frac{\partial X_n^{(0)}}{\partial v}=\sum_{i=1}^{n}\nu_i\xi_i',$$

denoting by ξ_i the components of the velocity of the origin, relative to the moving axes, when u only varies, and by ξ_i' those depending on v alone.

We find so again the following n equations connecting the rotations p, p' and the translations ξ, ξ',

$$\frac{\partial \xi_k}{\partial v}-\frac{\partial \xi_k'}{\partial u}=\sum_{i=1}^{k-1}(\xi_i' p_{ik}-\xi_i p_{ik}')-\sum_{i=k+1}^{n}(\xi_i' p_{ki}-\xi_i p_{ki}'), \qquad (12)$$
$$(k=1, 2, 3, \ldots\ldots, n).$$

18. We have also the $\frac{n(n-1)}{2}$ equations (11); thus: *For the general two-parametric displacements there exist* $\frac{n(n+1)}{2}$ *fundamental equations, viz.,* $\frac{n(n-1)}{2}$ *between the rotations only, and n between rotations and translations.*

19. Conversely, whenever the ξ's and the p's satisfy the two sets of fundamental equations (11) and (12), there exists a two-parametric displacement, and only one, with the rotations p and the translations ξ. The reasoning is the same as that of M. Darboux, "Surf.," I., p. 67.

---

20. Let us now consider the k-parametric displacements. It is obvious that we shall have $\frac{k(k-1)}{2}$ sets of fundamental equations formed like (11), for we can combine the k variables $u_1, u_2, \ldots\ldots, u_k$, upon which the moving system depends, in $\frac{k(k-1)}{2}$ ways two at a time. In the same way, we shall find, between rotations and translations, $\frac{k(k-1)}{2}$ sets formed like (12). Thus:

The general k-parametric displacement has $\frac{n(n+1)k(k-1)}{4}$ *fundamental equations.*

Letting $p^{(1)}$ be the rotations depending on u_1 alone, etc.; $p^{(k)}$ those depending on u_k alone, we shall have the same set (11) at first with $p^{(1)}, p^{(2)}$ (instead of p, p'), then with $p^{(1)}, p^{(3)}$, etc. And analogously for the equations (12). I omit writing all these equations, because they would occupy several pages.

21. These equations give, of course, *all* the relations existing between the p's (or the p's and ξ's), for those found between the p's and P's (or the p's, P's and ξ's) do not give *new* relations.

22. The projections, relatively to the moving axes, of the absolute velocity of a point M are

$$\left.\begin{array}{l}
\dfrac{dx_1}{dt}+\sum\limits_{i=1}^{k}\xi_1^{(i)}\dfrac{du_i}{dt}-x_2\sum\limits_{i=1}^{k}p_{12}^{(i)}\dfrac{du_i}{dt}-x_3\sum\limits_{i=1}^{k}p_{13}^{(i)}\dfrac{du_i}{dt}-\ldots\ldots-x_n\sum\limits_{i=1}^{k}p_{1n}^{(i)}\dfrac{du_i}{dt}, \\
\dfrac{dx_2}{dt}+\sum\limits_{i=1}^{k}\xi_2^{(i)}\dfrac{du_i}{dt}+x_1\sum\limits_{i=1}^{k}p_{12}^{(i)}\dfrac{du_i}{dt}-x_3\sum\limits_{i=1}^{k}p_{23}^{(i)}\dfrac{du_i}{dt}-\ldots\ldots-x_n\sum\limits_{i=1}^{k}p_{2n}^{(i)}\dfrac{du_i}{dt}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dfrac{dx_n}{dt}+\sum\limits_{i=1}^{k}\xi_n^{(i)}\dfrac{du_i}{dt}+x_1\sum\limits_{i=1}^{k}p_{1n}^{(i)}\dfrac{du_i}{dt}+x_2\sum\limits_{i=1}^{k}p_{2n}^{(i)}\dfrac{du_i}{dt}+\ldots\ldots+x_{n-1}\sum\limits_{i=1}^{k}p_{n-1\,n}^{(i)}\dfrac{du_i}{dt},
\end{array}\right\}(13)$$

where the $u_1, \ldots., u_k$ are supposed arbitrary functions of another variable t (the time for example).

---

23. Let us now consider the manifoldnesses of k dimensions of the space of n dimensions. Each of them has $\dfrac{k(k-1)n(n+1)}{4}$ fundamental kinematic equations between the rotations and translations of a system of axes $Mx_1 \ldots. x_n$ having as origin a point M of the manifoldness. The variables $u_1, \ldots. u_k$ must be, of course, all independent. We consider, further, the polyhedron $Mx_1x_2, \ldots., x_n$ in the following *special* position: The first k axes $x_1, x_2, \ldots., x_k$ lie in the *tangent* linear manifoldness of k dimensions (of the given manifoldness), and the other $n-k$ axes $x_{k+1}, \ldots., x_n$ lie in the *normal* linear $(n-k)$-dimensional manifoldness. It will then be

$$\xi_\sigma^{(1)}=\xi_\sigma^{(2)}=\ldots.=\xi_\sigma^{(k)}=0,$$
$$(\sigma=k+1, k+2, \ldots., n)$$

and the element of a curve lying on the k-dimensional manifoldness will be given by

$$\begin{aligned} ds^2 = {} & (\xi_1^{(1)}\, du_1 + \xi_1^{(2)}\, du_2 + \ldots\ldots + \xi_1^{(k)}\, du_k)^2 \\ & + (\xi_2^{(1)}\, du_1 + \xi_2^{(2)}\, du_2 + \ldots\ldots + \xi_2^{(k)}\, du_k)^2 \\ & + \ldots\ldots + (\xi_k^{(1)}\, du_1 + \xi_k^{(2)}\, du_2 + \ldots\ldots + \xi_k^{(k)}\, du_k)^2. \end{aligned}$$

The fundamental kinematic equations of the manifoldness become then a little shorter, but I omit writing them.

We can now put

$$\left.\begin{array}{l} \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)^2} = E_{11}, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)^2} = E_{22}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k)^2} = E_{kk}, \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)} \xi_\sigma^{(2)} = F_{12}, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)} \xi_\sigma^{(3)} = F_{23}, \ldots\ldots\ldots\ldots, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k-1)} \xi_\sigma^{(k)} = F_{k-1,k}, \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)} \xi_\sigma^{(3)} = F_{13}, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)} \xi_\sigma^{(4)} = F_{24}, \ldots\ldots, \quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k-2)} \xi_\sigma^{(k)} = F_{k-2,k}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)} \xi_\sigma^{(k)} = F_{1,k}; \end{array}\right\} \quad (14)$$

then

$$\begin{aligned} ds^2 = E_{11}\, du_1^2 + \ldots\ldots + E_{kk}\, du_k^2 + 2F_{12}\, du_1\, du_2 + \ldots\ldots + 2F_{1k}\, du_1\, du_k + {} & \\ + 2F_{23}\, du_2\, du_3 + \ldots\ldots + 2F_{2k}\, du_2\, du_k + \ldots\ldots + 2F_{k-1,\,k}\, du_{k-1}\, du_k, & \quad (15) \end{aligned}$$

with the discriminant

$$\Delta^2 \equiv \begin{vmatrix} E_{11} & F_{12} & F_{13} & \ldots\ldots\ldots\ldots & F_{1k} \\ F_{12} & E_{22} & F_{23} & \ldots\ldots\ldots\ldots & F_{2k} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ F_{1\,k-1} & F_{2\,k-1} & F_{3\,k-1} \ldots\ldots & E_{k-1\,k-1} & F_{k-1\,k} \\ F_{1k} & F_{2k} & F_{3k} \ldots\ldots & F_{k-1\,k} & E_{kk} \end{vmatrix}$$

$$= \begin{vmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \ldots\ldots & \xi_k^{(1)} \\ \xi_1^{(2)} & \xi_2^{(2)} & \ldots\ldots & \xi_k^{(2)} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \xi_1^{(k)} & \xi_2^{(k)} & \ldots\ldots & \xi_k^{(k)} \end{vmatrix}^2.$$

24. When the E's and F's are given (from the analytical equations of the manifoldness),* equations (14) serve to determine the ξ's as far as it is possible. For the complete determination of the ξ's, it is, of course, necessary to determine the position of the axes $x_1, \ldots, x_k$ in the tangent manifoldness, viz., to give $\frac{k(k-1)}{2}$ further relations between the ξ's, which, with the $\frac{k(k+1)}{2}$ relations (14), will entirely determine the k^2 ξ's. We now shall find the cosines and the rotations in terms of the E's, F's as far as it is possible.

25. We find the cosines by M. Darboux's method.

First we have

$$\left.\begin{array}{l}
\sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \alpha_\sigma = \frac{\partial x_1}{\partial u_1}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \beta_\sigma = \frac{\partial x_2}{\partial u_1}, \quad \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \nu_\sigma = \frac{\partial x_n}{\partial u_1}, \\
\sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \alpha_\sigma = \frac{\partial x_1}{\partial u_2}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \beta_\sigma = \frac{\partial x_2}{\partial u_2}, \quad \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \nu_\sigma = \frac{\partial x_n}{\partial u_2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \alpha_\sigma = \frac{\partial x_1}{\partial u_k}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \beta_\sigma = \frac{\partial x_2}{\partial u_k}, \quad \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \nu_\sigma = \frac{\partial x_n}{\partial u_k},
\end{array}\right\} \quad (16)$$

nk linear equations in the nk first cosines $\alpha_1, \ldots, \alpha_k$; $\beta_1, \ldots, \beta_k$; etc., $\nu_1, \ldots, \nu_k$. We find thus, Δ being the determinant of the ξ's,

$$\left.\begin{array}{l}
\alpha_1 = \frac{\left| \frac{\partial x_1}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \alpha_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_1}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots \\
\beta_1 = \frac{\left| \frac{\partial x_2}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \beta_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_2}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\nu_1 = \frac{\left| \frac{\partial x_n}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \nu_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_n}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots
\end{array}\right\} \quad (17)$$

($\tau = 1, 2, \ldots, k$ in the determinants).

---

*$E_{jj} = \sum_{i=1}^{n} \left(\frac{\partial x_i}{\partial u_j} \right)^2, \quad F_{jl} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial x_i}{\partial u_l}$.

26. There remain now the other $n(n-k)$ cosines; these are connected with themselves and with the former found nk cosines by $\frac{(n-k)(n+k+1)}{2}$ relations of the form of (2′), consequently $\frac{(n-k)(n-k-1)}{2}$ of these cosines remain indeterminate. And, indeed, in this general case, the position of the polyhedron of the manifoldness is not yet entirely determined. Assuming the ξ's to be all given, we only fix the position of the k axes lying in the tangent linear manifoldness, but the other $n-k$ axes lying in the normal linear manifoldness are not yet entirely determined. Only in the case of $k=n-1$, viz., of the manifoldness of the highest order, is the normal manifoldness a straight line and the cosines $\alpha_n, \beta_n, \ldots, \nu_n$ are entirely determined. In the general case of a k-dimensional manifoldness, the simplest manner to determine the axes in the normal manifoldness is the following: There exists a linear $(n-1)$-dimensional osculating manifoldness (or space) of the given manifoldness (very easy to show), and in this $(n-1)$-dimensional space there exists again a linear $(n-2)$-dimensional osculating space of the given manifoldness, etc., there exists, finally, a linear k-dimensional osculating manifoldness of the given manifoldness, the *tangent* manifoldness, lying in all the other osculating spaces of higher dimension. We now assume the axis x_n to be the normal to the osculating $(n-1)$-dimensional space; in this space, the axis x_{n-1} to be the normal to the osculating $(n-2)$-dimensional space, etc.; finally, the axis x_{k+1} to be normal in the osculating $(k+1)$-dimensional space to the k-dimensional osculating space.*

27. Assuming all the cosines to be so determined, we find the rotations in the following manner:

We have

$$\sum_{a,\,x} \alpha_{k+1}\, d\frac{\partial x_1}{\partial u_1} = \alpha_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\alpha_\sigma + \beta_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\beta_\sigma + \ \ldots\ + \nu_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\nu_\sigma,$$

---

* An example for $k<n-1$ is that of the curves in the ordinary space of three dimensions; the normal manifoldness is here the normal plane, and the cosines are entirely determined when we assume the axes y and z to be the *principal normal* and the *binormal* of the curve. This second straight line is the normal to the osculating plane, the first is the normal to the tangent, in this plane. A further example is the surface in the 4-dimensional space, etc. For $k=n-1$, an example constitutes the surface in the 3-dimensional space, the hypersurface in the 4-dimensional space, etc.

or

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} = \xi_1^{(1)} \sum \alpha_{k+1}\, d\alpha_1 + \xi_2^{(1)} \sum \alpha_{k+1}\, d\alpha_2 + \,\ldots\ldots + \xi_k^{(1)} \sum \alpha_{k+1}\, d\alpha_k,$$

viz.:

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} = \xi_1^{(1)}\, \Pi_{1\,k+1} + \,\ldots\ldots + \xi_k^{(1)}\, \Pi_{k\,k+1},$$

where

$$\Pi_{1\,k+1} = p^{(1)}_{1\,k+1}\, du_1 + p^{(2)}_{1\,k+1}\, du_2 + \,\ldots\ldots + p^{(k)}_{1\,k+1}\, du_k, \text{ etc.}$$

Further,

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_2} = \xi_1^{(2)}\, \Pi_{1\,k+1} + \xi_2^{(2)}\, \Pi_{2\,k+1} + \,\ldots\ldots + \xi_k^{(2)}\, \Pi_{k\,k+1},$$

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_3} = \xi_1^{(3)}\, \Pi_{1\,k+1} + \xi_2^{(3)}\, \Pi_{2\,k+1} + \,\ldots\ldots + \xi_k^{(3)}\, \Pi_{k\,k+1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_k} = \xi_1^{(k)}\, \Pi_{1\,k+1} + \xi_2^{(k)}\, \Pi_{2\,k+1} + \,\ldots\ldots + \xi_k^{(k)}\, \Pi_{k\,k+1},$$

and, consequently,

$$\Pi_{1\,k+1} = \frac{1}{\Delta} \begin{vmatrix} \sum\limits_{a,\,x} \alpha_{k+1}\, d\,\dfrac{\partial x_1}{\partial u_1} & \xi_2^{(1)} & \ldots\ldots & \xi_k^{(1)} \\ \sum\limits_{a,\,x} \alpha_{k+1}\, d\,\dfrac{\partial x_1}{\partial u_2} & \xi_2^{(2)} & \ldots\ldots & \xi_k^{(2)} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \sum\limits_{a,\,x} \alpha_{k+1}\, d\,\dfrac{\partial x_1}{\partial u_k} & \xi_2^{(k)} & \ldots\ldots & \xi_k^{(k)} \end{vmatrix},$$

(Δ being the determinant of the ξ's), etc.,

$$\Pi_{k\,k+1} = \frac{1}{\Delta} \begin{vmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \ldots\ldots & \xi_{k-1}^{(1)} & \sum\limits_{a,\,x} \alpha_{k+1}\, d\,\dfrac{\partial x_1}{\partial u_1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \xi_1^{(k)} & \xi_2^{(k)} & \ldots\ldots & \xi_{k-1}^{(k)} & \sum\limits_{a,\,x} \alpha_{k+1}\, d\,\dfrac{\partial x_1}{\partial u_k} \end{vmatrix}. \tag{18}$$

To find the $p^{(1)}_{1\,k+1}, p^{(2)}_{1\,k+1}, \ldots, p^{(k)}_{1\,k+1}; \ldots, p^{(1)}_{k\,k+1}, p^{(2)}_{k\,k+1}, \ldots, p^{(k)}_{k\,k+1}$, it is further only necessary to equate the coefficients of $du_1, du_2, \ldots, du_k$ on the left-hand and the right-hand side. We have thus expressed these p's in terms of the ξ's, the second derivatives of the x's and the $\alpha_{k+1}, \ldots, \nu_{k+1}$ *only* (their forms (4) contain also the $\alpha_1, \ldots, \alpha_k$, etc., $\nu_1, \ldots, \nu_k$).

We find in a quite analogous manner the

$$\begin{array}{l} \Pi_{1\,k+2}, \ldots, \Pi_{k\,k+2}, \\ \ldots\ldots\ldots\ldots\ldots \\ \Pi_{1n}, \quad \ldots, \Pi_{kn}, \end{array}$$

and, consequently, also the $p^{(1)}_{1\,k+2}, \ldots, p^{(k)}_{1\,k+2}; p^{(1)}_{2\,k+2}, \ldots, p^{(k)}_{2\,k+2}$; etc. $p^{(1)}_{k\,k+2}, \ldots, p^{(k)}_{k\,k+2}$, etc., etc. $p^{(1)}_{1n}, \ldots, p^{(k)}_{1n}$, etc., $p^{(1)}_{kn}, \ldots, p^{(k)}_{kn}$. All these p's we can find also in another way; it suffices to introduce in their expressions $p^{(1)}_{1\,k+1} = \sum \alpha_{k+1} \frac{\partial \alpha_1}{\partial u_1}$, etc., the values of $\frac{\partial \alpha_1}{\partial u_1}$, etc., from (17); the expressions so found, are, indeed, the same as those found from (18), etc., after some reductions.

28. Further,

$$\left.\begin{array}{l}
p^{(i)}_{k+1\,k+2} = \sum \alpha_{k+2} \frac{\partial \alpha_{k+1}}{\partial u_i}, \quad p^{(i)}_{k+1\,k+3} = \sum \alpha_{k+3} \frac{\partial \alpha_{k+1}}{\partial u_i}, \\
\qquad\qquad p^{(i)}_{k+1\,k+4} = \sum \alpha_{k+4} \frac{\partial \alpha_{k+1}}{\partial u_i}, \ldots, \quad p^{(i)}_{k+1\,n} = \sum \alpha_n \frac{\partial \alpha_{k+1}}{\partial u_i}, \\
p^{(i)}_{k+2\,k+3} = \sum \alpha_{k+3} \frac{\partial \alpha_{k+2}}{\partial u_i}, \quad p^{(i)}_{k+2\,k+4} = \sum \alpha_{k+4} \frac{\partial \alpha_{k+2}}{\partial u_i}, \ldots, \\
\qquad\qquad\qquad\qquad\qquad\qquad p^{(i)}_{k+2\,n} = \sum \alpha_n \frac{\partial \alpha_{k+2}}{\partial u_i}, \\
\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\qquad p^{(i)}_{n-1\,n} = \sum \alpha_n \frac{\partial \alpha_{n-1}}{\partial u_i}
\end{array}\right\} \quad (19)$$

($i = 1, 2, \ldots, k$). It is *impossible* to express these rotations by the ξ's, etc.

29. To find the last rotations, we use the formulæ

$$\sum_x \frac{\partial x_1}{\partial u_2}\cdot\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right)=\sum_{\sigma=1}^{k}\xi_\sigma^{(2)}\,\alpha_\sigma\left(\sum_{\sigma=1}^{k}\xi_\sigma^{(1)}\,\frac{\partial\alpha_\sigma}{\partial u_1}+\sum_{\sigma=1}^{k}\alpha_\sigma\,\frac{\partial\xi_\sigma^{(1)}}{\partial u_1}\right)+\cdots\cdots+$$

$$+\sum_{\sigma=1}^{k}\xi_\sigma^{(2)}\,\nu_\sigma\left(\sum_{\sigma=1}^{k}\xi_\sigma^{(1)}\,\frac{\partial\nu_\sigma}{\partial u_1}+\sum_{\sigma=1}^{k}\nu_\sigma\,\frac{\partial\xi_\sigma^{(1)}}{\partial u_1}\right)$$

$$\begin{aligned}
&=-\xi_1^{(2)}\,\xi_2^{(1)}p_{12}^{(1)}-\xi_1^{(2)}\,\xi_3^{(1)}p_{13}^{(1)}-\cdots\cdots-\xi_1^{(2)}\xi_k^{(1)}p_{1k}^{(1)}+\\
&\quad+\xi_2^{(2)}\,\xi_1^{(1)}p_{12}^{(1)}-\xi_2^{(2)}\,\xi_3^{(1)}p_{23}^{(1)}-\cdots\cdots-\xi_2^{(2)}\,\xi_k^{(1)}\,p_{2k}^{(1)}+\\
&\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+\\
&\quad+\xi_k^{(2)}\,\xi_1^{(1)}p_{1k}^{(1)}+\xi_k^{(2)}\,\xi_2^{(1)}p_{2k}^{(1)}+\cdots\cdots+\xi_k^{(2)}\,\xi_{k-1}^{(1)}p_{k-1k}^{(1)}+\cdots\cdots\\
&=\quad p_{12}^{(1)}\left(\xi_2^{(2)}\,\xi_1^{(1)}-\xi_1^{(2)}\xi_2^{(1)}\right)+p_{13}^{(1)}\left(\xi_3^{(2)}\,\xi_1^{(1)}-\xi_1^{(2)}\,\xi_3^{(1)}\right)\quad+\cdots\cdots+\\
&\quad+p_{k-1k}^{(1)}\left(\xi_k^{(2)}\,\xi_{k-1}^{(1)}-\xi_{k-1}^{(2)}\,\xi_k^{(1)}\right)+\xi_1^{(2)}\,\frac{\partial\xi_1^{(1)}}{\partial u_1}+\cdots\cdots+\xi_k^{(2)}\,\frac{\partial\xi_k^{(1)}}{\partial u_1}\,.
\end{aligned}$$

Forming, further, the similar expressions for

$$\sum_x \frac{\partial x_1}{\partial u_3}\,\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right),\ \cdots\cdots,\ \sum_x \frac{\partial x_1}{\partial u_k}\,\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right),$$

$$\sum_x \frac{\partial x_1}{\partial u_3}\,\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_2}\right),\ \cdots\cdots,\ \sum_x \frac{\partial x_1}{\partial u_k}\,\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_2}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_x \frac{\partial x_1}{\partial u_k}\,\frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_{k-1}}\right),$$

we have $\frac{k\,(k-1)}{2}$ linear equations for the $\frac{k\,(k-1)}{2}$ rotations $p_{12}^{(1)}, \ldots\ldots, p_{1k}^{(1)}, \ldots\ldots, p_{k-1k}^{(1)}$. And, putting in the expressions written above $\frac{\partial}{\partial u_2}, \ldots\ldots, \frac{\partial}{\partial u_k}$, instead of $\frac{\partial}{\partial u_1}$, we get the other rotations $p_{12}^{(2)}, \ldots\ldots, p_{1k}^{(2)}, \ldots\ldots, p_{k-1k}^{(2)}$, etc., etc., $p_{12}^{(k)}, \ldots\ldots, p_{1k}^{(k)}, \ldots\ldots, p_{k-1k}^{(k)}$. Thus for example :

$$p_{12}^{(1)} = \frac{1}{\Omega} \left| \begin{array}{llll}
\sum\limits_x \frac{\partial x_1}{\partial u_2} \frac{\partial^2 x_1}{\partial u_1^2} & \xi_3^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_3^{(1)} & \cdots\cdots & \xi_k^{(2)} \xi_{k-1}^{(1)} - \xi_{k-1}^{(2)} \xi_k^{(1)} \\
\sum\limits_x \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_1^2} & \xi_3^{(3)} \xi_1^{(1)} - \xi_1^{(3)} \xi_3^{(1)} & \cdots\cdots & \xi_k^{(3)} \xi_{k-1}^{(1)} - \xi_{k-1}^{(3)} \xi_k^{(1)} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \frac{\partial x_1}{\partial u_k} \frac{\partial^2 x_1}{\partial u_1^2} & \xi_3^{(k)} \xi_1^{(1)} - \xi_1^{(k)} \xi_3^{(1)} & \cdots\cdots & \xi_k^{(k)} \xi_{k-1}^{(1)} - \xi_{k-1}^{(k)} \xi_k^{(1)} \\
\sum\limits_x \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x}{\partial u_2 \partial u_1} & \xi_3^{(3)} \xi_1^{(2)} - \xi_1^{(3)} \xi_3^{(2)} & \cdots\cdots & \xi_k^{(3)} \xi_{k-1}^{(2)} - \xi_{k-1}^{(3)} \xi_k^{(2)} \\
\sum\limits_x \frac{\partial x_1}{\partial u_4} \frac{\partial^2 x}{\partial u_2 \partial u_1} & \xi_3^{(4)} \xi_1^{(2)} - \xi_1^{(4)} \xi_3^{(2)} & \cdots\cdots & \xi_k^{(4)} \xi_{k-1}^{(2)} - \xi_{k-1}^{(4)} \xi_k^{(2)} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \frac{\partial x_1}{\partial u_k} \frac{\partial^2 x}{\partial u_2 \partial u_1} & \xi_3^{(k)} \xi_1^{(2)} - \xi_1^{(k)} \xi_3^{(2)} & \cdots\cdots & \xi_k^{(k)} \xi_{k-1}^{(2)} - \xi_{k-1}^{(k)} \xi_k^{(2)} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \frac{\partial x_1}{\partial u_k} \frac{\partial^2 x}{\partial u_{k-1} \partial u_1} & \xi_3^{(k)} \xi_1^{(k-1)} - \xi_1^{(k)} \xi_3^{(k-1)} & \cdots\cdots & \xi_k^{(k)} \xi_{k-1}^{(k-1)} - \xi_{k-1}^{(k)} \xi_k^{(k-1)}
\end{array} \right|$$

+ an expression of only the ξ's. (Ω being the determinant of the system.)

And after some reductions we get

$$p_{12}^{(1)} = \frac{1}{2\Omega} \left| \begin{array}{lll}
-\frac{1}{2} \frac{\partial E_{11}}{\partial u_2} + \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(1)} \frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & & \xi_3^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_3^{(1)} \cdots\cdots \\
-\frac{1}{2} \frac{\partial E_{11}}{\partial u_3} + \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(1)} \frac{\partial \xi_\sigma^{(3)}}{\partial u_1} & & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
-\frac{1}{2} \frac{\partial E_{11}}{\partial u_k} + \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(1)} \frac{\partial \xi_\sigma^{(k)}}{\partial u_1} & & \cdots\cdots \\
\frac{1}{2} \left(\frac{\partial F_{23}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_3} + \frac{\partial F_{13}}{\partial u_2} \right) & - \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(3)} \frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & \cdots\cdots \\
\frac{1}{2} \left(\frac{\partial F_{24}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_4} + \frac{\partial F_{14}}{\partial u_2} \right) & - \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(4)} \frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\frac{1}{2} \left(\frac{\partial F_{2k}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_k} + \frac{\partial F_{1k}}{\partial u_2} \right) & - \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(k)} \frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\frac{1}{2} \left(\frac{\partial F_{k-1\,k}}{\partial u_1} - \frac{\partial F_{1\,k-1}}{\partial u_k} + \frac{\partial F_{1k}}{\partial u_{k-1}} \right) & - \sum\limits_{\sigma=1}^{k} \xi_\sigma^{(k)} \frac{\partial \xi_\sigma^{(k-1)}}{\partial u_1} & \cdots\cdots
\end{array} \right| ; \quad (20)$$

the $p_{12}^{(1)}$ (and, consequently, also the $p_{13}^{(1)}, \ldots, p_{k-1k}^{(1)}$) can be expressed by only the ξ's and the E's and F's, analogously to the rotations, in the case of the surfaces of the 3-dimensional space. And all the other rotations $p_{12}^{(i)}, \ldots,$ $p_{1k}^{(i)}, \ldots, p_{k-1k}^{(i)}$ can also be expressed by the same quantities. Thus, for example,

$$p_{12}^{(3)} = \frac{1}{\Omega}\left|\begin{array}{llll}
\frac{1}{2}\frac{\partial E_{22}}{\partial u_1} - \sum\limits_{\sigma=1}^{k}\xi_\sigma^{(2)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & & \xi_3^{(2)}\xi_1^{(1)} - \xi_1^{(2)}\xi_3^{(1)} & \cdots \\
\frac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_3} + \frac{\partial F_{13}}{\partial u_2}\right) & -\sum\limits_{\sigma=1}^{k}\xi_\sigma^{(3)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(3)}\xi_1^{(1)} - \xi_1^{(3)}\xi_3^{(1)} & \cdots \\
\frac{1}{2}\left(\frac{\partial F_{24}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_4} + \frac{\partial F_{14}}{\partial u_2}\right) & -\sum\limits_{\sigma=1}^{k}\xi_\sigma^{(4)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(4)}\xi_1^{(1)} - \xi_1^{(4)}\xi_3^{(1)} & \cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\
\frac{1}{2}\left(\frac{\partial F_{2k}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_k} + \frac{\partial F_{1k}}{\partial u_2}\right) & -\sum\limits_{\sigma=1}^{k}\xi_\sigma^{(k)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(k)}\xi_1^{(1)} - \xi_1^{(k)}\xi_3^{(1)} & \cdots \\
-\frac{1}{2}\frac{\partial E_{22}}{\partial u_3} + \sum\limits_{\sigma=1}^{k}\xi_\sigma^{(2)}\frac{\partial \xi_\sigma^{(3)}}{\partial u_2} & & \xi_3^{(3)}\xi_1^{(2)} - \xi_1^{(3)}\xi_3^{(2)} & \cdots \\
-\frac{1}{2}\frac{\partial E_{22}}{\partial u_4} + \sum\limits_{\sigma=1}^{k}\xi_\sigma^{(2)}\frac{\partial \xi_\sigma^{(4)}}{\partial u_2} & & \xi_3^{(4)}\xi_1^{(2)} - \xi_1^{(4)}\xi_3^{(2)} & \cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\
-\frac{1}{2}\frac{\partial E_{22}}{\partial u_k} + \sum\limits_{\sigma=1}^{k}\xi_\sigma^{(2)}\frac{\partial \xi_\sigma^{(k)}}{\partial u_2} & & \xi_3^{(k)}\xi_1^{(2)} - \xi_1^{(k)}\xi_3^{(2)} & \cdots \\
\frac{1}{2}\left(\frac{\partial F_{34}}{\partial u_2} - \frac{\partial F_{23}}{\partial u_4} + \frac{\partial F_{24}}{\partial u_3}\right) & -\sum\limits_{\sigma=1}^{k}\xi_\sigma^{(4)}\frac{\partial \xi_\sigma^{(3)}}{\partial u_2} & \xi_3^{(4)}\xi_1^{(3)} - \xi_1^{(4)}\xi_3^{(3)} & \cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\
\frac{1}{2}\left(\frac{\partial F_{k-1k}}{\partial u_2} - \frac{\partial F_{2\,k-1}}{\partial u_k} + \frac{\partial F_{2k}}{\partial u_{k-1}}\right) & -\sum\limits_{\sigma=1}^{k}\xi_\sigma^{(k)}\frac{\partial \xi_\sigma^{(k-1)}}{\partial u_2} & \xi_3^{(k)}\xi_1^{(k-1)} - \xi_1^{(k)}\xi_3^{(k-1)} & \cdots
\end{array}\right|$$

etc., etc.

30. If the *parametric* lines:

$u_1 = \text{const.}, u_2 = \text{const.}, \ldots, u_{k-1} = \text{const.}; u_1 = \text{const.}, u_2 = \text{const.}, \ldots, u_{k-2}$
$= \text{const.}, u_k = \text{const.}; \ldots\ldots; u_2 = \text{const.}, \ldots, u_k = \text{const.};$

are the *lines of curvature* of the given manifoldness and the axes $x_k, \ldots, x_1$ coincide with their tangents, the formulæ become a little shorter; but I shall omit this point and examine the case of $k = n - 1$, viz., the case of the manifoldnesses, which, in our general space, correspond exactly to the surfaces of the space of three dimensions.

31. Now

$$\left.\begin{array}{l} \displaystyle\sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\alpha_\sigma = \frac{\partial x_1}{\partial u_1}, \quad \sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\beta_\sigma = \frac{\partial x_2}{\partial u_1}, \ldots\ldots, \sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\nu_\sigma = \frac{\partial x_n}{\partial u_1}, \\ \ldots,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \displaystyle\sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\alpha_\sigma = \frac{\partial x_1}{\partial u_{n-1}}, \quad \sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\beta_\sigma = \frac{\partial x_2}{\partial u_{n-1}}, \ldots\ldots, \sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\nu_\sigma = \frac{\partial x_n}{\partial u_{n-1}}, \end{array}\right\} \quad (22)$$

from these equations we have at once

$$\left.\begin{array}{l} \alpha_1 = \dfrac{\begin{vmatrix} \dfrac{\partial x_1}{\partial u_\tau} \xi_2^{(\tau)} \xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix}}{D}, \ldots\ldots, \alpha_{n-1} = \dfrac{\begin{vmatrix} \xi_1^{(\tau)} \xi_2^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_1}{\partial u_\tau} \end{vmatrix}}{D}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \nu_1 = \dfrac{\begin{vmatrix} \dfrac{\partial x_n}{\partial u_\tau} \xi_2^{(\tau)} \xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix}}{D}, \ldots\ldots, \nu_{n-1} = \dfrac{\begin{vmatrix} \xi_1^{(\tau)} \xi_2^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_n}{\partial u_\tau} \end{vmatrix}}{D}, \end{array}\right\} \quad (23)$$

D being the determinant of the system.

32. The other n cosines $\alpha_n, \beta_n, \ldots\ldots, \nu_n$ are given by a well-known property of the orthogonal determinants *under the text*;* namely:

$$\alpha_n = \pm \begin{vmatrix} \beta_1 \ldots\ldots \beta_{n-1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \nu_1 \ldots\ldots \nu_{n-1} \end{vmatrix} (-1)^{n-1}, \ldots\ldots, \nu_n = \pm \begin{vmatrix} \alpha_1 \ldots\ldots \alpha_{n-1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \mu_1 \ldots\ldots \mu_{n-1} \end{vmatrix},$$

each element equals its *algebraical complement* multiplied by ± 1 (the determinant). It is thus (if we suppose the determinant, $= +1$, which is always possible):

$$\alpha_n = \frac{(-1)^{n-1}}{D^{n-1}} \left\| \begin{array}{llll} \begin{vmatrix} \dfrac{\partial x_2}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \begin{vmatrix} \xi_1^{(\tau)} \dfrac{\partial x_2}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \ldots\ldots & \begin{vmatrix} \xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_2}{\partial u_\tau} \end{vmatrix} \\ \begin{vmatrix} \dfrac{\partial x_3}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \begin{vmatrix} \xi_1^{(\tau)} \dfrac{\partial x_3}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \ldots\ldots & \begin{vmatrix} \xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_3}{\partial u_\tau} \end{vmatrix} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ \begin{vmatrix} \dfrac{\partial x_{n-1}}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \begin{vmatrix} \xi_1^{(\tau)} \dfrac{\partial x_{n-1}}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \ldots\ldots & \begin{vmatrix} \xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_{n-1}}{\partial u_\tau} \end{vmatrix} \\ \begin{vmatrix} \dfrac{\partial x_n}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \begin{vmatrix} \xi_1^{(\tau)} \dfrac{\partial x_n}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)} \end{vmatrix} & \ldots\ldots & \begin{vmatrix} \xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)} \dfrac{\partial x_n}{\partial u_\tau} \end{vmatrix} \end{array} \right\|;$$

---

[*] See, for example, Pascal's I Determinanti, p. 205.

but this *compound* determinant is only equal to the product

$$\begin{vmatrix} \dfrac{\partial x_2}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_2}{\partial u_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial u_{n-1}} \end{vmatrix} . D^{n-2}.*$$

Consequently,

$$\alpha_n = \frac{1}{D} \frac{d(x_2, x_3, \ldots, x_n)}{d(u_1, u_2, \ldots, u_{n-1})} (-1)^{n-1}, \quad \beta_n = \frac{1}{D} \frac{d(x_1, x_3, \ldots, x_{nn})}{d(u_1, u_2, \ldots, u_{n-1})} (-1)^{n-2},$$

$$\ldots, \nu_n = \frac{1}{D} \frac{d(x_1, x_2, \ldots, x_{n-1})}{d(u_1, u_2, \ldots, u_{n-1})}. \qquad (24)$$

33. It remains to find the rotations in terms of the ξ's, E's and F's. We get for this purpose from (22),

$$\sum_{\alpha, \pi} \alpha_n d \frac{\partial x_1}{\partial u_1} = \alpha_n \Big[\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\alpha_\sigma + \sum_{\sigma=1}^{n-1} \alpha_\sigma d\xi_\sigma^{(1)} \Big] + \beta_n \Big[\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\beta_\sigma + \sum_{\sigma=1}^{n-1} \beta_\sigma d\xi_\sigma^{(1)} \Big] +$$

$$+ \ldots + \nu_n \Big[\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\nu_\sigma + \sum_{\sigma=1}^{n-1} \nu_\sigma d\xi_\sigma^{(1)} \Big],$$

---

* This theorem, not yet observed, I think, can be easily shown. Taking the determinants A and B and Γ,

$$A \equiv \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix}, \qquad B \equiv \begin{vmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{vmatrix},$$

$$\Gamma \equiv \left\| \begin{matrix} \begin{vmatrix} \beta_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} & \cdots & \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1\,n-1} & \beta_{11} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \alpha_{n\,n-1} & \beta_{n1} \end{vmatrix} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \begin{vmatrix} \beta_{1n} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{nn} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} & \cdots & \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1\,n-1} & \beta_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \alpha_{n\,n-1} & \beta_{nn} \end{vmatrix} \end{matrix} \right\|,$$

the determinant Γ can be written

$$\Gamma = \begin{vmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{vmatrix} \begin{vmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{vmatrix},$$

A_{ik} being the *algebraical complements* of the α_{ik}'s (in the determinant A); and by a well-known theorem we have consequently (Pascal, p. 48)

$$\Gamma = B . A^{n-1}. \qquad \text{Q. E. D.}$$

or

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} \sum\limits_{i=1}^{n-1}\frac{\partial^2 x_1}{\partial u_1\,\partial u_i}du_i & \sum\limits_{i=1}^{n-1}\frac{\partial^2 x_2}{\partial u_1\,\partial u_i}du_i & \cdots\cdots & \sum\limits_{i=1}^{n-1}\frac{\partial^2 x_n}{\partial u_1\,\partial u_i}du_i \\ \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \cdots\cdots & \frac{\partial x_n}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \cdots\cdots & \frac{\partial x_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial u_{n-1}} & \frac{\partial x_2}{\partial u_{n-1}} & \cdots\cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}$$

$$=\xi_1^{(1)}\Pi_{1n}+\xi_2^{(1)}\Pi_{2n}+\cdots\cdots+\xi_{n-1}^{(1)}\Pi_{n-1n},$$

where $\Pi_{1n}=p_{1n}^{(1)}\,du_1+\cdots\cdots+p_{1n}^{(n-1)}\,du_{n-1}$, etc.

Also

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} \sum\limits_{i=1}^{n-1}\frac{\partial^2 x_1}{\partial u_2\,\partial u_i}du_i & \cdots\cdots & \sum\limits_{i=1}^{n-1}\frac{\partial^2 x_n}{\partial u_2\,\partial u_i}du_i \\ \frac{\partial x_1}{\partial u_1} & \cdots\cdots & \frac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial u_{n-1}} & \cdots\cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}$$

$$=\xi_1^{(2)}\Pi_{1n}+\xi_2^{(2)}\Pi_{2n}+\cdots\cdots+\xi_{n-1}^{(2)}\Pi_{n-1n},$$

and other $n-3$ equations with $\partial u_3,\ \xi^{(3)};\ \partial u_4,\ \xi^{(4)};\ \cdots\cdots;\ \partial u_{n-1},\ \xi^{(n-1)}$. Solving them, we get

$$\Pi_{1n}=\frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \overline{D}_1 & \xi_2^{(1)} & \cdots\cdots & \xi_{n-1}^{(1)} \\ \overline{D}_2 & \xi_2^{(2)} & \cdots\cdots & \xi_{n-1}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{D}_{n-1} & \xi_2^{(n-1)} & \cdots\cdots & \xi_{n-1}^{(n-1)} \end{vmatrix},\ \cdots\cdots,$$

$$\Pi_{n-1n}=\frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \cdots\cdots & \xi_{n-2}^{(1)} & \overline{D}_1 \\ \xi_1^{(2)} & \cdots\cdots & \xi_{n-2}^{(2)} & \overline{D}_2 \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1^{(n-1)} & \cdots\cdots & \xi_{n-2}^{(n-1)} & \overline{D}_{n-1} \end{vmatrix},$$

and, consequently,

$$\left.\begin{aligned}
p_{1n}^{(1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} D_{11} & \xi_2^{(1)} & \dots & \xi_{n-1}^{(1)} \\ D_{21} & \xi_2^{(2)} & \dots & \xi_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ D_{n-11} & \xi_2^{(n-1)} & \dots & \xi_{n-1}^{(n-1)} \end{vmatrix}, \dots, \\
p_{1n}^{(n-1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} D_{1\,n-1} & \xi_2^{(1)} & \dots & \xi_{n-1}^{(1)} \\ D_{2\,n-1} & \xi_2^{(2)} & \dots & \xi_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ D_{n-1\,n-1} & \xi_2^{(n-1)} & \dots & \xi_{n-1}^{(n-1)} \end{vmatrix}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
p_{n-1n}^{(1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \dots & \xi_{n-2}^{(1)} & D_{11} \\ \dots & \dots & \dots & \dots \\ \xi_1^{(n-1)} & \dots & \xi_{n-2}^{(n-1)} & D_{n-11} \end{vmatrix}, \dots, \\
p_{n-1n}^{(n-1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \dots & \xi_{n-2}^{(1)} & D_{1\,n-1} \\ \dots & \dots & \dots & \dots \\ \xi_1^{(n-1)} & \dots & \xi_{n-2}^{(n-1)} & D_{n-1\,n-1} \end{vmatrix},
\end{aligned}\right\} \quad (25)$$

where

$$D_{ik} = \begin{vmatrix} \dfrac{\partial^2 x_1}{\partial u_i\,\partial u_k} & \dots & \dfrac{\partial^2 x_n}{\partial u_i\,\partial u_k} \\ \dfrac{\partial x_1}{\partial u_1} & \dots & \dfrac{\partial x_n}{\partial u_1} \\ \dots & \dots & \dots \\ \dfrac{\partial x_1}{\partial u_{n-1}} & \dots & \dfrac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}.$$

34. Further,

$$\sum_{\kappa} \frac{\partial x_1}{\partial u_2}\, d\,\frac{\partial x_1}{\partial u_1} = \sum_{\sigma=1}^{n-1} \xi_\sigma^{(2)}\,\alpha_\sigma \left(\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)}\,d\alpha_\sigma + \sum_{\sigma=1}^{n-1} \alpha_\sigma\,d\xi_\sigma^{(1)}\right) + \dots + \sum_{\sigma=1}^{n-1} \xi_\sigma^{(2)}\,\nu_\sigma \left(\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)}\,d\nu_\sigma + \sum_{\sigma=1}^{n-1} \nu_\sigma\,d\xi_\sigma^{(1)}\right),$$

or

$$\begin{aligned}
&\left(\frac{\partial F_{12}}{\partial u_1} - \tfrac{1}{2}\frac{E_{11}}{\partial u_2}\right) du_1 + \tfrac{1}{2}\frac{\partial E_{22}}{\partial u_1} du_2 + \tfrac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1} - \frac{\partial F_{13}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_3}\right) du_3 + \\
&\quad + \tfrac{1}{2}\left(\frac{\partial F_{24}}{\partial u_1} - \frac{\partial F_{14}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_4}\right) du_4 + \ldots\ldots + \tfrac{1}{2}\left(\frac{\partial F_{2\,n-1}}{\partial u_1} - \frac{\partial F_{1\,n-1}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_{n-1}}\right) du_{n-1} \\
&\quad = \xi_1^{(2)} d\xi_1^{(1)} + \xi_2^{(2)} d\xi_2^{(1)} + \ldots\ldots + \xi_{n-1}^{(2)} d\xi_{n-1}^{(1)} + \\
&\quad + \xi_1^{(2)}\left(-\xi_2^{(1)}\Pi_{12} - \xi_3^{(1)}\Pi_{13} - \ldots\ldots - \xi_{n-1}^{(1)}\Pi_{1\,n-1}\right) + \\
&\quad + \xi_2^{(2)}\left(+\xi_1^{(1)}\Pi_{12} - \xi_3^{(1)}\Pi_{23} - \ldots\ldots - \xi_{n-1}^{(1)}\Pi_{2\,n-1}\right) + \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&\quad + \xi_{n-1}^{(2)}\left(\xi_1^{(1)}\Pi_{1\,n-1} + \xi_2^{(1)}\Pi_{2\,n-1} + \ldots\ldots + \xi_{n-2}^{(1)}\Pi_{n-2\,n-1}\right) \\
&\quad = \sum_{\sigma=1}^{n-1}\xi_\sigma^{(2)} d\xi_\sigma^{(1)} + \Pi_{12}\left(\xi_2^{(2)}\xi_1^{(1)} - \xi_1^{(2)}\xi_2^{(1)}\right) + \\
&\quad + \Pi_{13}\left(\xi_{(3)}^{(2)}\xi_{(1)}^{(1)} - \xi_1^{(2)}\xi_3^{(1)}\right) + \ldots\ldots + \Pi_{1\,n-1}\left(\xi_{n-1}^{(2)}\xi_1^{(1)} - \xi_1^{(2)}\xi_{n-1}^{(1)}\right) + \\
&\quad + \Pi_{23}\left(\xi_3^{(2)}\xi_2^{(1)} - \xi_2^{(2)}\xi_3^{(1)}\right) + \ldots\ldots + \Pi_{2\,n-1}\left(\xi_{n-1}^{(2)}\xi_2^{(1)} - \xi_2^{(2)}\xi_{n-1}^{(1)}\right) + \\
&\qquad\qquad\qquad\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&\qquad\qquad\qquad\qquad\qquad + \Pi_{n-2\,n-1}\left(\xi_{n-1}^{(2)}\xi_{n-2}^{(1)} - \xi_{n-2}^{(2)}\xi_{n-1}^{(1)}\right).
\end{aligned}$$

Forming, further, the analogous expressions

$$\begin{aligned}
&\sum_x \frac{\partial x_1}{\partial u_3} d\frac{\partial x_1}{\partial u_1}, \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_{n-1}} d\frac{\partial x_1}{\partial u_1}, \\
&\sum_x \frac{\partial x_1}{\partial u_3} d\frac{\partial x_1}{\partial u_2}, \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_{n-1}} d\frac{\partial x_1}{\partial u_2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\sum_x \frac{\partial x_1}{\partial u_{n-1}} d\frac{\partial x_1}{\partial u_{n-2}},
\end{aligned}$$

and solving these $\frac{(n-1)(n-2)}{2}$ equations, we find the rotations $p_{12}^{(i)}, \ldots\ldots,$ $p_{1\,n-1}^{(i)}$; $p_{23}^{(i)}. \ldots\ldots, p_{2\,n-1}^{(i)}$; etc., $p_{n-2\,n-1}^{(i)}$ (taking the coefficients of $du_1, du_2, \ldots\ldots,$ du_{n-1} on the left-hand and the right-hand side):

$$p_{12}^{(1)}=\frac{1}{\Omega}\left|\begin{array}{llll}
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_2}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial\xi_\sigma^{(2)}}{\partial u_1} & \cdots & \xi_3^{(2)}\xi_1^{(1)} & -\xi_1^{(2)}\xi_3^{(1)}\\
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_3}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial\xi_\sigma^{(3)}}{\partial u_1} & & \xi_3^{(3)}\xi_1^{(1)} & -\xi_1^{(3)}\xi_3^{(1)}\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_{n-1}}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial\xi_\sigma^{(n-1)}}{\partial u_1} & & \xi_3^{(n-1)}\xi_1^{(1)} & -\xi_1^{(n-1)}\xi_3^{(1)}\\
\frac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1}-\frac{\partial F_{12}}{\partial u_3}+\frac{\partial F_{13}}{\partial u_2}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(3)}\frac{\partial\xi_\sigma^{(2)}}{\partial u_1} & & \xi_3^{(3)}\xi_1^{(2)} & -\xi_1^{(3)}\xi_3^{(2)}\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
\frac{1}{2}\left(\frac{\partial F_{n-2\,n-1}}{\partial u_1}-\frac{\partial F_{1\,n-2}}{\partial u_{n-1}}+\frac{\partial F_{1\,n-1}}{\partial u_{n-2}}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial\xi_\sigma^{(n-2)}}{\partial u_1} & & \xi_3^{(n-1)}\xi_1^{(n-2)} & -\xi_1^{(n-1)}\xi_3^{(n-2)}
\end{array}\right|$$

etc., et , etc.

$$p_{12}^{(n-1)}=\frac{1}{\Omega}\left|\begin{array}{llll}
\frac{1}{2}\left(\frac{\partial F_{2\,n-1}}{\partial u_1}+\frac{\partial F_{12}}{\partial u_{n-1}}-\frac{\partial F_{1\,n-1}}{\partial u_2}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(2)}\frac{\partial\xi_\sigma^{(1)}}{\partial u_{n-1}} & & \xi_3^{(2)}\xi_1^{(1)} & -\xi_1^{(2)}\xi_3^{(1)}\\
\frac{1}{2}\left(\frac{\partial F_{3\,n-1}}{\partial u_1}+\frac{\partial F_{13}}{\partial u_{n-1}}-\frac{\partial F_{1\,n-1}}{\partial u_3}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(3)}\frac{\partial\xi_\sigma^{(1)}}{\partial u_{n-1}} & & \cdots\cdots & \cdots\cdots\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_1}-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial\xi_\sigma^{(1)}}{\partial u_{n-1}} & & \cdots\cdots & \cdots\cdots\\
\frac{1}{2}\left(\frac{\partial F_{3\,n-1}}{\partial u_2}+\frac{\partial F_{23}}{\partial u_{n-1}}-\frac{\partial F_{2\,n-1}}{\partial u_3}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(3)}\frac{\partial\xi_\sigma^{(2)}}{\partial u_{n-1}} & & \cdots\cdots & \cdots\cdots\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_2}-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial\xi_\sigma^{(2)}}{\partial u_{n-1}} & & \cdots\cdots & \cdots\cdots\\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_{n-2}}-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial\xi_\sigma^{(n-2)}}{\partial u_{n-1}} & & \cdots\cdots & \cdots\cdots
\end{array}\right|$$

etc., et , etc.

35. Let us finally consider the case, where the parametric lines

$$u_1=\text{const.},\ u_2=\text{const.},\ \ldots\ldots,\ u_{n-2}=\text{const.};\ u_1=\text{const.},\ \ldots\ldots,\ u_{n-3}=\text{const.},$$
$$u_{n-1}=\text{const.};\ \ldots\ldots\ldots;\ u_2=\text{const.},\ \ldots\ldots,\ u_{n-1}=\text{const.}$$

are the *lines of curvature* and the axes $x_{n-1}, \ldots, x_1$ coincide with their tangents. Then

$$\left.\begin{aligned} F_{12} = F_{13} = \ldots &= F_{1\,n-1} = \\ = F_{23} = \ldots &= F_{2\,n-1} = \\ = \ldots &= \\ &= F_{n-2\,n-1} = 0\,; \\ D_{12} = D_{13} = \ldots &= D_{1\,n-1} = \\ = D_{23} = \ldots &= D_{2\,n-1} = \\ = \ldots &= \\ &= D_{n-2\,n-1} = 0, \end{aligned}\right\} \tag{27}$$

and

$$\left.\begin{aligned} &\xi_2^{(1)} = \xi_3^{(1)} = \ldots = \xi_{n-1}^{(1)} = \\ = &\xi_1^{(2)} = \xi_3^{(2)} = \ldots = \xi_{n-1}^{(2)} = \\ = &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \\ = &\xi_1^{(n-1)} = \xi_2^{(n-1)} = \ldots = \xi_{n-2}^{(n-1)} = 0, \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} \xi_1^{(1)} &= \sqrt{E_{11}}, \\ \xi_2^{(2)} &= \sqrt{E_{22}}, \\ &\ldots\ldots\ldots \\ \xi_{n-1}^{(n-1)} &= \sqrt{E_{n-1\,n-1}}. \end{aligned}\right\} \tag{28'}$$

36. We get, further, from (25),

$$\left.\begin{aligned} p_{1n}^{(2)} &= p_{1n}^{(3)} &&= \ldots = p_{1n}^{(n-1)} = 0, \\ p_{2n}^{(1)} &= p_{2n}^{(3)} &&= \ldots = p_{2n}^{(n-1)} = 0, \\ p_{3n}^{(1)} &= p_{3n}^{(2)} = p_{3n}^{(4)} &&= \ldots = p_{3n}^{(n-1)} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{n-1n}^{(1)} &= p_{n-1n}^{(2)} = p_{n-1n}^{(3)} &&= \ldots = p_{n-1n}^{(n-2)} = 0, \end{aligned}\right\} \tag{29}$$

$(n-2)(n-1)$ of the first set of rotations are equal zero.

37. The other rotations of this set are given by the formulæ (25),

$$\left.\begin{aligned} p_{1n}^{(1)} &= D_{11}\xi_2^{(2)}\xi_3^{(3)}\ldots\ldots\ldots\xi_{n-1}^{(n-1)}\,\frac{(-1)^{n-1}}{D^2} = D_{11}\,\frac{1}{\sqrt{E_{11}}}\,\frac{(-1)^{n-1}}{D}\,, \\ p_{2n}^{(2)} &= D_{22}\,\xi_1^{(1)}\xi_3^{(3)}\ldots\ldots\ldots\xi_{n-1}^{(n)}\,\frac{(-1)^{n-1}}{D^2} = D_{22}\,\frac{1}{\sqrt{E_{22}}}\cdot\frac{(-1)^{n-1}}{D}\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{n-1n}^{(n-1)} &= D_{n-1\,n-1}\,\xi_1^{(1)}\,\xi_2^{(2)}\ldots\xi_{n-2}^{(n-2)}\,.\,\frac{(-1)^{n-1}}{D^2} = \\ &= D_{n-1\,n-1}\,\frac{1}{\sqrt{E_{n-1\,n-1}}}\cdot\frac{(-1)^{n-1}}{D}\,, \end{aligned}\right\} \tag{30}$$

38. Finally from (26),

$$\left.\begin{aligned}
p^{(1)}_{12} &= -\tfrac{1}{2}\frac{\partial E_{11}}{\partial u_2}\,\frac{1}{\xi^{(2)}_2\,\xi^{(1)}_1} = -\frac{1}{\xi^{(2)}_2}\,\frac{\partial \xi^{(1)}_1}{\partial u_2} = -\frac{1}{2\sqrt{E_{11}E_{22}}}\,\frac{\partial E_{11}}{\partial u_2},\\
p^{(1)}_{13} &= -\frac{1}{\xi^{(3)}_3}\,\frac{\partial \xi^{(1)}_1}{\partial u_3} = \frac{-1}{2\sqrt{E_{11}E_{33}}}\,\frac{\partial E_{11}}{\partial u_3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(1)}_{1\,n-1} &= -\frac{1}{\xi^{(n-1)}_{n-1}}\,\frac{\partial \xi^{(1)}_1}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{11}E_{n-1\,n-1}}}\cdot\frac{\partial E_{11}}{\partial u_{n-1}},\\
p^{(1)}_{23} &= 0,\\
p^{(1)}_{24} &= 0,\\
&\cdots\cdots\\
p^{(1)}_{2\,n-1} &= 0,\\
&\cdots\cdots\\
p^{(1)}_{n-2\,n-1} &= 0.
\end{aligned}\right\}\quad(31)$$

$$\left.\begin{aligned}
p^{(2)}_{12} &= +\frac{1}{\xi^{(1)}_1}\,\frac{\partial \xi^{(2)}_2}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}E_{22}}}\,\frac{\partial E_{22}}{\partial u_1},\\
p^{(2)}_{13} &= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(2)}_{1\,n-1} &= 0.\\
p^{(2)}_{23} &= -\frac{1}{\xi^{(3)}_3}\,\frac{\partial \xi^{(2)}_2}{\partial u_3} = \frac{-1}{2\sqrt{E_{22}E_{33}}}\cdot\frac{\partial E_{22}}{\partial u_3},\\
p^{(2)}_{24} &= -\frac{1}{\xi^{(4)}_4}\,\frac{\partial \xi^{(2)}_2}{\partial u_4} = \frac{-1}{2\sqrt{E_{22}E_{44}}}\cdot\frac{\partial E_{22}}{\partial u_4},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(2)}_{2\,n-1} &= -\frac{1}{\xi^{(n-1)}_{n-1}}\,\frac{\partial \xi^{(2)}_2}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{22}E_{n-1\,n-1}}}\,\frac{\partial E_{22}}{\partial u_{n-1}},\\
p^{(2)}_{34} &= 0,\\
&\cdots\cdots\\
p^{(2)}_{n-2\,n-1} &= 0.
\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}
p^{(3)}_{12} &= 0,\\
p^{(3)}_{13} &= +\frac{1}{\xi^{(1)}_1}\,\frac{\partial \xi^{(3)}_3}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}E_{33}}}\,\frac{\partial E_{33}}{\partial u_1},\\
p^{(3)}_{14} &= 0,\\
&\cdots\cdots\\
p^{(3)}_{1\,n-1} &= 0,\\
p^{(3)}_{23} &= +\frac{1}{\xi^{(2)}_2}\,\frac{\partial \xi^{(3)}_3}{\partial u_2} = \frac{+1}{2\sqrt{E_{22}E_{33}}}\,\frac{\partial E_{33}}{\partial u_2},
\end{aligned}\right\}\quad(33)$$

$$\left.\begin{aligned}
p_{24}^{(3)} &= 0,\\
&\cdots\cdots\\
p_{2\,n-1}^{(3)} &= 0,\\
p_{34}^{(3)} &= \frac{-1}{\xi_4^{(4)}}\,\frac{\partial \xi_3^{(3)}}{\partial u_4} = \frac{-1}{2\sqrt{E_{33}\,E_{44}}}\,\frac{\partial E_{33}}{\partial u_4},\\
p_{35}^{(3)} &= \frac{-1}{\xi_5^{(5)}}\,\frac{\partial \xi_3^{(3)}}{\partial u_5} = \frac{-1}{2\sqrt{E_{33}\,E_{55}}}\cdot\frac{\partial E_{33}}{\partial u_5},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
p_{3\,n-1}^{(3)} &= \frac{-1}{\xi_{n-1}^{(n-1)}}\cdot\frac{\partial \xi_3^{(3)}}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{33}\,E_{n-1\,n-1}}}\,\frac{\partial E_{33}}{\partial u_{n-1}},\\
p_{45}^{(3)} &= 0,\\
p_{46}^{(3)} &= 0,\\
&\cdots\cdots\\
p_{n-2\,n-1}^{(3)} &= 0.
\end{aligned}\right\}\ \text{(Continued.)}\ (33)$$

$$\left.\begin{aligned}
&\cdots\cdots\\
&\cdots\cdots\\
&\cdots\cdots
\end{aligned}\right\}\ (34)$$

$$\left.\begin{aligned}
p_{12}^{(n-1)} &= 0,\\
p_{13}^{(n-1)} &= 0,\\
&\cdots\cdots\\
p_{1\,n-2}^{(n-1)} &= 0,\\
p_{1\,n-1}^{(n-1)} &= \frac{+1}{\xi_1^{(1)}}\,\frac{\partial \xi_{n-1}^{(n-1)}}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}\,E_{n-1\,n-1}}}\cdot\frac{\partial E_{n-1\,n-1}}{\partial u_1},\\
p_{23}^{(n-1)} &= 0,\\
&\cdots\cdots\\
p_{2\,n-2}^{(n-1)} &= 0,\\
p_{2\,n-1}^{(n-1)} &= \frac{+1}{\xi_2^{(2)}}\,\frac{\partial \xi_{n-1}^{(n-1)}}{\partial u_2} = \frac{+1}{2\sqrt{E_{22}\,E_{n-1\,n-1}}}\,\frac{\partial E_{n-1\,n-1}}{\partial u_2}.\\
p_{34}^{(n-1)} &= 0,\\
&\cdots\cdots\\
p_{3\,n-2}^{(n-1)} &= 0,\\
p_{3\,n-1}^{(n-1)} &= \frac{+1}{\xi_3^{(3)}}\,\frac{\partial \xi_{n-1}^{(n-1)}}{\partial u_3} = \frac{+1}{2\sqrt{E_{33}\,E_{n-1\,n-1}}}\,\frac{\partial E_{n-1\,n-1}}{\partial u_3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
p_{n-2\,n-1}^{(n-1)} &= \frac{+1}{\xi_{n-2}^{(n-2)}}\,\frac{\partial \xi_{n-1}^{(n-1)}}{\partial u_{n-2}} = \frac{+1}{2\sqrt{E_{n-2\,n-2}\,E_{n-1\,n-1}}}\,\frac{\partial E_{n-1\,n-1}}{\partial u_{n-2}};
\end{aligned}\right\}\ (35)$$

the p's, which are not zero, are those which have two equal *indices.*[*]

---

[*] In the paper of Prof. Craig is an erratum: In the formula (66), (p. 155) the multipliers of the

39. It is now obvious that we may stud the lines, the surfaces, etc., on the manifoldness of the $n-1$ dimensions (or that of k generally) in the same manner as M. Darboux does.

GÖTTINGEN (Germany), *July*, 1899.

---

brackets must be $\zeta''\zeta\zeta'$ and not $\xi''\xi$, ξ', and, consequ ntly, the formulæ of the page 156 must be written (with the p_{13}, p'_{13}, p''_{13} of Professor Craig)

$$\begin{cases} p''_{12}=0, & p'_{12}=\frac{+1}{2\sqrt{E_{11}\,E_{2}}}\cdot\frac{\partial E_{22}}{\partial t}, \quad p_{12}=\frac{-1}{2\sqrt{E_{11}\,E_{22}}}\,\frac{\partial E_{11}}{\partial u}, \\ p_{13}=\frac{+1}{2\sqrt{E_{33}\,E_{11}}}\cdot\frac{\partial E'_{11}}{\partial v}, \quad p'_{13}=0, & p''_{13}=\frac{-1}{2\sqrt{E_{33}\,E_{11}}}\,\frac{\partial E_{33}}{\partial t}, \\ p''_{23}=\frac{+1}{2\sqrt{E_{22}\,E_{33}}}\,\frac{\partial E_{33}}{\partial u}, \quad p'_{23}=\frac{-1}{2\sqrt{E_{22}\,E_{3}}}\,\frac{\partial E_{22}}{\partial v}, & p_{23}=0. \end{cases}$$

On the Product of Two Substitutions.

By G. A. Miller.

The main object of this paper is to prove the following

Theorem.—*If l, m, n are any three integers greater than unity, of which we call the greatest k, it is always possible to find three substitutions (L, M, N) of $k+2$ or some smaller number of elements and of orders l, m, n respectively such that $LM = N$.*

The following lemmas may be employed to prove this theorem:

Lemma I. *If two circular substitutions (S_1, S_2) have an odd number of consecutive elements in common, while all their other elements are distinct, these common elements may be so arranged that the product $S_1 S_2$ is a circular substitution which involves all the elements of S_1 and S_2 except an arbitrary even number of the common elements.*

Let

$S_1 = (a_1 \dots a_{2\alpha+1} \dots a_{2n+1} \dots a_k)$[*] and

$$S_2 = (a_{2\alpha+1} \dots a_1 a_{2\alpha+2} \dots a_{2n+1} a_{k+1} \dots a_l).\dagger$$

The product $S_1 S_2$ will contain: 1st, all the common elements that have odd subscripts larger than 2α; 2nd, all the elements of S_1 that are not contained in S_2; 3rd, all the common elements that have even subscripts larger than 2α; 4th, all the elements of S_2 that are not contained in S_1. Hence

$$S_1 S_2 = (a_{2\alpha+1} a_{2\alpha+3} \dots a_{2n+1} a_{2n+2} \dots a_k a_{2\alpha+2} a_{2\alpha+4} \dots a_{2n} a_{k+1} \dots a_l).$$

It is clear that the number (2α) of the common elements that do not appear in this product may be made to vary from 0 to $2n$.

---

[*] Throughout this article it is supposed that the omitted elements have for subscripts all the consecutive numbers in order, between the given limits, unless the contrary is indicated by the elements which are written out.

[†] When $\alpha = n$, the elements $a_{2\alpha+2} \dots a_{2n+1}$ do not occur.

Lemma II. *If two circular substitution (S_1, S_2) have an even number of consecutive elements in common while all their o ier elements are distinct, these common elements may be so arranged that the produc $S_1 S_2$ is a circular substitution which involves all the elements of S_1 and S_2 except a y arbitrary odd number of the common elements.*

Let

$$S_1 = (a_1 \ldots\ldots a_{2a} \ldots\ldots a_{2n} \ldots\ldots a_k) \text{ and } S_2 = (a_{2a} \ldots\ldots a_1 a_{2a+1} \ldots\ldots a_{2n} a_{k+1} \ldots\ldots a_l).$$

The product $S_1 S_2$ will contain: 1st, all he common elements that have even subscripts larger than $2a - 1$; 2nd, all the elements of S_1 that are not contained in S_2; 3rd, all the common elements that ave odd subscripts larger than $2a$; 4th, all the elements of S_2 that are not co tained in S_1. Hence

$$S_1 S_2 = (a_{2a} a_{2a+2} \ldots\ldots a_{2n} a_{2n+1} \ldots\ldots a_k \, {}_{2a+1} a_{2a+3} \ldots\ldots a_{2n-1} a_{k+1} \ldots\ldots a_l).$$

When $2a = 2n = k = l$, the number of ele nents in the product $S_1 S_2$ should be one according to the lemma. Such a subs itution is clearly impossible, the corresponding product being identity. Fro these two lemmas we observe that the elements of two circular substitutions (S_1, S_2) may be so chosen that their product is a circular substitution whose or er is any arbitrary number that may be obtained by diminishing the sum of the rders of S_1 and S_2 by any odd positive number which is less than twice the small r one of these two orders.

Lemma III. *If a circular substitution (S_1) has an odd number (greater than unity) of elements in common with a substit tion (S_2) which is composed of a transposition and a circular substitution, the com on elements may be so arranged that the product $S_1 S_2$ is a circular substitution whic contains all the elements of S_1 and S_2 except any arbitrary odd number (less than he total number) of the common elements.*

Let

$$S_1 = (a_1 \ldots\ldots a_{2a+2} \ldots\ldots a_{2n+1} \ldots\ldots a_k) \text{ a d}$$

$$S_2 = (a_{2a+2} a_{2a+3})(a_{2a+1} \ldots a_1 a_{2a+4} \ldots\ldots a_{2n+1} a_{k+1} \ldots\ldots a_l).^*$$

The product $S_1 S_2$ will contain: 1st, all the common elements that have odd subscripts larger than $2a$; 2nd, all the el ments of S_1 that are not contained in

---

* The elements $a_{2a+4} \ldots\ldots a_{2n+1}$ are omitted wh n $2a + 3$ has its maximal value, viz., $2n + 1$.

S_2; 3rd, all the common elements that have even subscripts larger than $2\alpha + 2$; 4th, all the elements of S_2 that are not contained in S_1. Hence

$$S_1 \dot{S}_2 = (a_{2\alpha+1} a_{2\alpha+3} \dots a_{2n+1} a_{2n+2} \dots a_k a_{2\alpha+4} a_{2\alpha+6} \dots a_{2n} a_{k+1} \dots a_l).$$

The common elements which do not occur in $S_1 S_2$ are clearly $a_1 \dots a_{2\alpha}$ and $a_{2\alpha+2}$. When there is only one such element, it is a_2, α being equal to zero.

Lemma IV. *If a circular substitution* (S_1) *has an even number (greater than zero) of elements in common with a substitution* (S_2) *which is composed of a transposition and a circular substitution, the common elements may be so arranged that the product* $S_1 S_2$ *is a circular substitution which contains all the elements of* S_1 *and* S_2 *except an arbitrary even number (less than the total number) of the common elements.*

We shall first suppose that the number of common elements $(2n)$ exceeds two, and that the elements which do not occur in $S_1 S_2$ are $a_1 \dots a_{2\alpha-1}$ and $a_{2\alpha+1}$ $(\alpha \neq 0)$. It is clear that $\alpha + 1 \not> n$.

Let

$$S_1 = (a_1 \dots a_{2\alpha} \dots a_{2n} \dots a_k) \text{ and}$$
$$S_2 = (a_{2\alpha+1} a_{2\alpha+2})(a_{2\alpha} \dots a_1 a_{2\alpha+3} \dots a_{2n} a_{k+1} \dots a_l).^*$$

The product $S_1 S_2$ will contain: 1st, all the common elements that have even subscripts greater than $2\alpha - 1$; 2nd, all the elements of S_1 that are not contained in S_2; 3rd, all the common elements that have odd subscripts greater than $2\alpha + 1$; 4th, all the elements of S_2 that are not contained in S_1. Hence

$$S_1 S_2 = (a_{2\alpha} a_{2\alpha+2} \dots a_{2n} a_{2n+1} \dots a_k a_{2\alpha+3} a_{2\alpha+5} \dots a_{2n-1} a_{k+1} \dots a_l).$$

If all the common elements are to appear in $S_1 S_2$, we may use the following substitutions:

$$S_1 = (a_1 \dots a_{2n} \dots a_k) \text{ and } S_2 = (a_1 a_3)(a_2 a_4 a_5 \dots a_{2n} a_{k+1} \dots a_l).$$

Finally, if S_1 and S_2 have only two common elements, they may have the following forms:

$$S_1 = (a_2 a_1 a_3 a_4 \dots a_k) \text{ and } S_2 = (a_2 a_{k+1})(a_1 a_{k+2} \dots a_l).$$

From the last two lemmas we observe that the elements of two substitutions (one (S_1) being circular and the other, (S_2), consisting of a transposition and a circular substitution) may be so chosen that the product $S_1 S_2$ is a circular substitution whose order is any arbitrary number that may be obtained by diminishing the sum of the degrees of S_1 and S_2 by an even positive integer (excluding 0) which is less than twice the smaller of these two degrees. If the degree of S_2 is supposed to be even, we may deduce the following result from these lemmas: The elements of S_1 and S_2 may be so chosen that their product is a circular substitution whose order is any arbitrary integer that may be obtained by diminishing the sum of the orders of S_1 and S_2 by any even positive integer (including 0) which does not exceed twice the smaller of these orders. Combining this result with the one stated after Lemma II, we have the following: If one of two arbitrary integers is even, it is always possible to find two substitutions whose orders are these integers respectively and the sum of whose degrees does not exceed the sum of their orders by more than two, such that their product is a circular substitution whose order is any arbitrary number that may be obtained by diminishing the sum of the two given integers by any integer (0 included) which does not exceed twice the smaller of the given integers.

---

* The elements $a_{2\alpha+3} \dots a_{2n}$ do not occur when $\alpha + 1 = n$.

We are now prepared to prove the given theorem without much difficulty. Since the relation $LM=N$ is equivalent to $MN^{-1}=L^{-1}$, as well as to $N^{-1}L=M^{-1}$, we may assume $l\lesseqgtr m\lesseqgtr n$. This will be done in what follows. Hence we have

$$n=\alpha m+\beta=\alpha m+k(l-1)+\varepsilon,\quad m=\gamma l+\delta,\quad \delta<l,\quad \varepsilon<l-1,\quad \beta<m.$$

We shall first consider the case when l is even, and represent M as follows:

$$M=(a_1\ldots\ldots a_m)(a_{m+1}\ldots\ldots a_{2m})\ldots\ldots(\;)\ldots\ldots(a_{(\alpha-1)m+1}\ldots\ldots a_{\alpha m})\,.$$

When $\alpha>1$, we let

$$L=(a_1a_{\alpha m+1})(a_2a_{\alpha m+2})\ldots\ldots(\;)\ldots\ldots(a_\beta a_{\alpha m+\beta})(a_m a_{m+1})(a_{2m}a_{2m+1})\ldots\ldots(\;)\ldots\ldots$$
$$(a_{(\alpha-1)m}a_{(\alpha-1)m+1}\ldots\ldots a_{(\alpha-1)m+l-1}).$$

When $\alpha=1$ and $\beta>l$ we let

$$L=(a_1a_{m+1})(a_2a_{m+2})\ldots\ldots(\;)\ldots\ldots(a_{\beta-l+1}a_{m+\beta-l+1})(a_m a_{m+\beta-l+2}\ldots\ldots a_{m+\beta}).$$

When $\alpha=1$ and $\beta\lesseqgtr l$, the proper substitutions were determined above. In what follows we may, therefore, confine our attention to the cases when l is an odd number. We shall assume, unless the contrary is stated, that M has always the given value and confine our attention to the determination of suitable values for L.

When α is even and $\varepsilon=1$, we may employ the given M, and the following L_1 for L:

$$\begin{aligned} L_1 = (a_1\, a_{\alpha m+1} \ldots\ldots a_{\alpha m+l-1})(a_2\, a_{\alpha m+l} \ldots\ldots a_{\alpha m+2(l-1)}) \ldots\ldots (\;) \ldots\ldots \\ (a_k\, a_{\alpha m+(k-1)(l-1)+1} \ldots\ldots a_{\alpha m+k(l-1)})(a_m\, a_{\alpha m+\beta}\, a_{m+1}\, a_{m+2} \ldots\ldots a_{m+l-2}) \\ (a_{2m}\, a_{2m+1}\, a_{3m+1} \ldots\ldots a_{3m+l-2}) \ldots\ldots (\;) \ldots\ldots \\ (a_{(\alpha-2)m}\, a_{(\alpha-2)m+1}\, a_{(\alpha-1)m+1} \ldots\ldots a_{(\alpha-1)m+l-2}). \end{aligned}$$

When α is even and ε is odd and greater than 1, we multiply L_1 by the cycle $(a_{k+1} \ldots\ldots a_{k+l-\varepsilon+1}\, a_{\alpha m+k(l-1)+1} \ldots\ldots a_{\alpha m+\beta-1})$ and use the product for L. When α and ε are even, we replace the element $a_{\alpha m+\beta}$ in L_1 by the element $a_{\alpha m+\beta+1}$ and multiply the result by the cycle

$$(a_{k+2}\, a_{\alpha m+\beta+2}\, a_{k+1}\, a_{k+3} \ldots\ldots a_{k+l-\varepsilon-1}\, a_{\alpha m+k(l-1)+1} \ldots\ldots a_{\alpha m+\alpha}.)^{*}$$

This product is to be used for L. It remains to consider the cases when both α and l are odd. When $\varepsilon = 0$ (α and l being odd), we employ the following L_2 for L in the equation $LM = N$:

$$\begin{aligned} L_2 = (a_1\, a_{\alpha m+1} \ldots\ldots a_{\alpha m+l-1})(a_2\, a_{\alpha m+l} \ldots\ldots a_{\alpha m+2(l-1)}) \ldots\ldots (\;) \ldots\ldots \\ (a_k\, a_{\alpha m+(k-1)(l-1)+1} \ldots\ldots a_{\alpha m+k(l-1)})(a_m\, a_{m+1}\, a_{2m+1} \ldots\ldots a_{2m+l-2}) \\ (a_{3m}\, a_{3m+1}\, a_{4m+1} \ldots\ldots a_{4m+l-2}) \ldots\ldots (\;) \ldots\ldots \\ (a_{(\alpha-2)m}\, a_{(\alpha-2)m+1}\, a_{(\alpha-1)m+1} \ldots\ldots a_{(\alpha-1)m+l-2}). \end{aligned}$$

If ε is even and greater than 0, we may obtain a suitable value for L by multiplying L_2 by the cycle $(a_{k+1} \ldots\ldots a_{k+l-\varepsilon}\, a_{\alpha m+k(l-1)+1} \ldots\ldots a_{\alpha m+\beta})$. When each of the three numbers α, l, ε is odd while m is even, we may use the following M_3, L_3 for M and L respectively:

$$M_3 = (a_1 \ldots\ldots a_m)(a_{m+1} \ldots\ldots a_{2m}) \ldots\ldots (\;) \ldots\ldots (a_{(\alpha-1)m+1} \ldots\ldots a_{\alpha m})(a_{\alpha m+1}\, a_{\alpha m+2}),$$
$$L_3 = L_2 \text{ multiplied by } (a_{k+1} \ldots\ldots a_{k+l-\varepsilon+1}\, a_{\alpha m+2} \ldots\ldots a_{\alpha m+\beta}).$$

When each of the four numbers α, l, ε, m is odd and $\varepsilon < l - 2$, we may employ the following M_4, L_4 for M and L respectively:

$$M_4 = M,$$
$$\begin{aligned} L_4 = L_2 \text{ multiplied by the cycle } (a_{k+2}\, a_{\alpha m+\beta+2}\, a_{k+1}\, a_{k+3} \ldots\ldots \\ a_{k+l-\varepsilon-2}\, a_{\alpha m+k(l-1)+1} \ldots\ldots a_{\alpha m+\beta+1}).^{\dagger} \end{aligned}$$

When each of the four numbers α, l, ε, m is odd, $l - 2 = \varepsilon$ and $m > l$, we may employ the following M_5, L_5 for M and L respectively:

---

[*] When $l = 3$, this cycle consists of its first three elements, when $\varepsilon = 0$ the elements $a_{\alpha m+k(l-1)+1} \ldots a_{\alpha m+\beta}$ are to be omitted.

[†] When $l = \varepsilon + 4$, the elements $a_{k+1} \ldots\ldots a_{k+l-\varepsilon-2}$ are to be omitted.

$M_5 = M$,

$L_5 = L_2$ multiplied by the two cycles $(a_{k+1}\,a_{\alpha}{}_{+k(l-1)+1} \ldots\ldots a_{\alpha m+\beta+1})$

$(a_{k+3}\,a_{\alpha m+\beta+2}\,a_{k+2}\,a_{k+4} \ldots\ldots a_{k+l})$.*

Finally, when the four numbers α, l, ε, n are odd and $l - 2 = \varepsilon = m - 2$, we may employ the following M_6, L_6 for M and L respectively:

$$M_6 = (a_1 \ldots\ldots a_m)(a_{m+1} \ldots\ldots a_{2m}) \ldots\ldots (\;) \ldots\ldots (a_{\alpha m+1} \ldots\ldots a_{(\alpha+1)m}),$$

$$L_6 = (a_m\,a_{m+1}\,a_{2m+1} \ldots\ldots a_{3m-2})(a_{3m}\,a_{3m+1}\,a_{4m+1} \ldots\ldots a_{5m-2}) \ldots\ldots (\;) \ldots\ldots$$

$$(a_{(\alpha-2)m}\,a_{(\alpha-2)m+1}\,a_{(\alpha-1)m+1} \ldots\ldots a_{\alpha m-2})({}_{\alpha m}\,a_{\alpha m+3}\,a_{\alpha m+1}\,a_{\alpha m+4} \ldots\ldots a_{(\alpha+1)m}).\dagger$$

We have now found suitable substitutions in each of the possible cases, and we never employed more than $n + 2$ different elements. Hence, the proof of the given theorem is complete. There is clearly no substitution except identity whose order is less than two. This was the reason for assuming that each of the numbers l, m, n is greater than unity. With respect to abstract-group theory the given theorem may be stated as follows: It is possible to find three operators whose orders are any three arbitrary numbers greater than unity such that the product of two of them is the third operator. Hence, we can choose in a triply infinite number of ways the orders of three operators which satisfy the condition that one of them is the product of the other two. While an operator of a finite order always generates a cyclical group of a finite order it follows directly from the given theorem that two operators of any given finite orders greater than unity, which are otherwise unrestricted, generate a group of an infinite order that contains operators of every possible finite order.

CORNELL UNIVERSITY.

---

* When $l = 3$, this cycle consists of its first three elements.

† When $m = 3$, this cycle contains only its first three elements.

On Continuous Binary Linearoid Groups, and the Corresponding Differential Equations and Λ Functions.

BY E. J. WILCZYNSKI.

In a former paper we have shown that, corresponding to every group of the form

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x;\ a_1, \ldots a_r)\, y_k, \tag{1}$$

where the r parameters a_i are essential, there exists a system of differential equations of order r, whose general solutions are given by (1), if $y_1, \ldots y_n$ form a fundamental system. The functions ϕ_{ik} were supposed to be uniform functions of x, and it was found that, if the parameters a_i were properly chosen, ϕ_{ik} were uniform functions of the parameters also.

We propose, in this paper, to discuss these groups, the corresponding differential equations, and their solutions for the case that $n = 2$.

It may, of course, happen that ϕ_{ik} are all independent of x, so that the group becomes linear and $r \leq 4$. Thus, the linear group appears as a special case of the linearoid. By using Lie's types of linear groups, the types of linearoid groups could be more easily found than by the methods of this paper. But the methods employed here, and which are also all essentially due to Lie, are more elementary and furnish not only the *types* of all groups investigated, but these groups themselves. Moreover, a number of interesting formulæ and theorems are thus found which, even for linear groups, escape notice when the *type* method is used.

§1.—*One-parameter Groups. Case I.*

Let

$$Uf = (\psi_{11}(x)y_1 + \psi_{12}(x)y_2)\frac{\partial f}{\partial y_1} + (\psi_{21}(x)y_1 + \psi_{22}(x)y_2)\frac{\partial f}{\partial y_2} \tag{2}$$

be the infinitesimal transformation of the group. Its finite transformations are found by integrating the simultaneous system

$$\frac{d\eta_i}{dt} = \psi_{i1} y_1 + \psi_{i2} y_2, \qquad (i = 1,\ 2) \qquad (3)$$

with the initial conditions $\eta_i = y_i$ for $t = 0$.

Denoting, therefore, by ρ_1, ρ_2, the roots of the quadratic

$$| \psi_{ik} - \delta_{ik} \rho | = 0, \qquad (\delta_{ii} = 1,\ \delta_{ik} = 0,\ i \neq k), \quad (4)$$

and determining λ_1 and λ_2 from the equations

$$\left.\begin{aligned} \lambda_i (\psi_{11} - \rho_i) + \psi_{12} &= 0, \\ \lambda_i \psi_{21} + \psi_{22} - \rho_i &= 0, \end{aligned}\right\} \qquad (i = 1,\ 2) \qquad (5)$$

the finite equations of the group are found to be

$$\left.\begin{aligned} \eta_1 &= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1 e^{\rho_1 t} - \lambda_2 e^{\rho_2 t}) y_1 - \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{\rho_1 t} - e^{\rho_2 t}) y_2, \\ \eta_2 &= \frac{1}{\lambda_1 - \lambda_2} (e^{\rho_1 t} - e^{\rho_2 t}) y_1 - \frac{1}{\lambda_1 - \lambda_2} (\lambda_2 e^{\rho_1 t} - \lambda_1 e^{\rho_2 t}) y_2, \end{aligned}\right\} \qquad (6)$$

where λ_1 and λ_2, and, therefore, ρ_1 and ρ_2 must be distinct. We treat then under Case I, the case where the characteristic equation of the infinitesimal transformation of the group has distinct roots. The case of equal roots will be considered as Case II.

Our group is of the required form, i. e., its coefficients are uniform functions of x, if

$$D = (\psi_{11} - \psi_{22})^2 + 4 \psi_{12} \psi_{21} \qquad (7)$$

is the square of a uniform function of x.

The invariant of the group can be found without integration by noting the fact that there exist two relative invariants.

For from (6) we find easily

$$\left.\begin{aligned} \eta_1 - \lambda_1 \eta_2 &= e^{\rho_2 t} (y_1 - \lambda_1 y_2), \\ \eta_1 - \lambda_2 \eta_2 &= e^{\rho_1 t} (y_1 - \lambda_2 y_2). \end{aligned}\right\} \qquad (8)$$

If, then, we write

$$Y_i = y_1 - \lambda_i y_2, \quad H_i = \eta_1 - \lambda_i \eta_2, \qquad (i = 1,\ 2), \qquad (9)$$

Y_1, Y_2 are relative invariants, and

$$\vartheta_1 = Y_1^{\rho_1} Y_2^{-\rho_2} = H_1^{\rho_1} H_2^{-\rho_2} \tag{10}$$

is an absolute invariant.

We are interested in the differential invariants of the first order. According to (8), we have the following equations:

$$\frac{d}{dx}\,\frac{1}{\rho_j}\log H_i = \frac{d}{dx}\,\frac{1}{\rho_j}\log Y_i, \qquad (i, j = 1,\ 2;\ i \neq j).$$

We have, therefore, the differential invariants:

$$\vartheta_2 = \frac{Y_1'}{Y_1} - \frac{\rho_2'}{\rho_2}\log Y_1, \quad \vartheta_3 = \frac{Y_2'}{Y_2} - \frac{\rho_1'}{\rho_1}\log Y_2, \tag{11}$$

which are obviously independent. ϑ_1, ϑ_2, ϑ_3 are the three independent invariants of the original group G extended once. The relation holds:

$$\frac{1}{\vartheta_1}\,\frac{d\vartheta_1}{dx} = \frac{1}{\rho_1\rho_2}\,\frac{d(\rho_1\rho_2)}{dx}\log\vartheta_1 + \rho_1\vartheta_2 - \rho_2\vartheta_3. \tag{12}$$

Let us consider a system of differential equations

$$\Phi_k(\vartheta_1, \vartheta_2, \vartheta_3) = 0, \qquad (k = 1,\ 2,\ 3),$$

where Φ_k are three independent functions of ϑ_1, ϑ_2, ϑ_3 with coefficients depending only upon x, but such that if, by solving these equations, we find

$$\vartheta_k = f_k(x), \qquad (k = 1,\ 2,\ 3),$$

the equation

$$\frac{1}{f_1}\,\frac{df_1}{dx} = \frac{1}{\rho_1\rho_2}\,\frac{d(\rho_1\rho_2)}{dx}\log f_1 + \rho_1 f_2 - \rho_2 f_3 \tag{13}$$

will be identically satisfied. Then this system, or in full,

$$\left.\begin{aligned}
\frac{d\log}{dx}(y_1 - \lambda_1 y_2) - \frac{d\log\rho_2}{dx}\log(y_1 - \lambda_1 y_2) &= f_2(x),\\
\frac{d\log}{dx}(y_1 - \lambda_2 y_2) - \frac{d\log\rho_1}{dx}\log(y_1 - \lambda_2 y_2) &= f_3(x),\\
(y_1 - \lambda_1 y_2)^{\rho_1}(y_1 - \lambda_2 y_2)^{-\rho_2} \qquad &= f_1(x),
\end{aligned}\right\} \tag{14}$$

where (13) also is verified is to be considered.

However, in place of the last equation (14), it is often better to write

$$\rho_1 \log (y_1 - \lambda_1 y_2) - \rho_2 \log (y_1 - \lambda_2 y_2) = \log f_1(x) = f(x), \quad f_1(x) = e^{f(x)}. \quad (15)$$

For the left member of this equation is a more characteristic invariant of our group than $Y_1^{\rho_1} Y_2^{-\rho_2} = \vartheta_1$. For instance, ϑ_1 is invariant not only for the transformations (8) or

$$H_1 = e^{\rho_2 t} Y_1, \quad H_2 = e^{\rho_1 t} Y_2,$$

but also for all transformations of the form

$$H_1 = e^{\sigma_2 t_k} Y_1; \quad H_2 = e^{\sigma_1 t_k} Y_2,$$

provided that the relation is possible

$$(\rho_1 \sigma_2 - \rho_2 \sigma_1) t_k = 2k\pi i,$$

where k is any integer. Thus, ϑ_1 may admit, besides all of the transformations of our group, an infinite but discreet number of substitutions, i. e., ϑ_1 may admit altogether a mixed group.

If, now, y_1, y_2 is a fundamental system of our equations, the general solutions η_1, η_2 will be given by equations (6), where t is an arbitrary constant. If λ_1, λ_2; ρ_1, ρ_2; f_2; f_3; f_1 or f are uniform functions of x, the functions y_1, y_2 will have the following property: They are uniform, finite and continuous, except for certain values of x, say $a_1, a_2, \ldots.$, and whenever the variable x describes a circuit around a_i, y_1, y_2 undergo a substitution A_i contained in the group (6) or some other substitution, leaving the left members of (14) unaltered, provided that there exists such a substitution. We shall leave such exceptional cases aside and prove only that the quantities still at our disposal can be so chosen that the points a_i and the substitutions A_i belonging to them can be arbitrarily assigned, all of the substitutions A_i being of form (6).

If y_1, y_2 are functions of the character just described, obviously f_1, f_2, f_3, f must be uniform functions of x. Let them, then, be uniform functions, and study directly the differential equations.

The integration of the first two equations (14) gives

$$\left.\begin{aligned} \log Y_1 &= \frac{\rho_2}{\rho_2^0}\left[C + \rho_2^0 \int_{x_0}^{x} \frac{f_2(x)}{\rho_2}\, dx\right], \\ \log Y_2 &= \frac{\rho_1}{\rho_1^0}\left[C' + \rho_1^0 \int_{x_0}^{x} \frac{f_3(x)}{\rho_1}\, dx\right], \end{aligned}\right\} \quad (16)$$

where ρ_i^0 denotes the value of ρ_i for $x = x_0$, an arbitrary value of x. But the two constants C and C' are not independent. For, according to the third equation (14), we must have

$$Y_1^{\rho_1} Y_2^{-\rho_2} = e^{\rho_1 \rho_2 \left(\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0}\right) + \rho_1 \rho_2 \int_{x_0}^{x} \left(\frac{f_2(x)}{\rho_2} - \frac{f_3(x)}{\rho_1}\right) dx} = f_1(x), \tag{17}$$

or, if we use (13), to simplify the exponential,

$$e^{\rho_1 \rho_2 \left(\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0}\right) + \log f_1(x) - \frac{\rho_1 \rho_2}{\rho_1^0 \rho_2^0} \log f_1(x_0)} = f_1(x).$$

Therefore,

$$\rho_1 \rho_2 \left[\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0} - \frac{1}{\rho_1^0 \rho_2^0} \log f_1(x_0)\right] = 2k\pi i$$

must be an integral multiple of $2\pi i$. If $\rho_1 \rho_2$ is not a constant, we conclude at once

$$\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0} - \frac{1}{\rho_1^0 \rho_2^0} \log f_1(x_0) = 0, \tag{18}$$

Let $a_1 \ldots . a_m$ be the poles of $\frac{f_2(x)}{\rho_2}$ and $\frac{f_3(x)}{\rho_1}$, so that

$$\frac{f_2(x)}{\rho_2} = \sum_{k=1}^{m} \frac{c_k}{x - a_k} + \ldots . , \quad \frac{f_3(x)}{\rho_1} = \sum_{k=1}^{m} \frac{d_k}{x - a_k} + \ldots . , \tag{19}$$

where the terms not written contain no terms of the form $\frac{1}{x - \lambda}$, and, therefore, give uniform functions on integration.

But, according to (13), we have

$$f_1(x) = e^{\rho_1 \rho_2 \left[a + \int \left(\frac{f_2}{\rho_2} - \frac{f_3}{\rho_1}\right) dx\right]} \tag{20}$$

a being an arbitrary constant. But $f_1(x)$ must be a uniform function of x. If $\rho_1 \rho_2$ is not a constant, this requires $c_k = d_k$, *so that*

$$\frac{f_2(x)}{\rho_2} - \frac{f_3(x)}{\rho_1},$$

when expanded in the vicinity of a_k *contains no term of the form* $(x - a_k)^{-1}$. If $\rho_1 \rho_2 = \gamma =$ const., the only thing that follows is that $(c_k - d_k)\gamma$ must be an integer. However, in that case, we will consider equation (15) in place of the third equation (14), i. e., we require $f(x) = \log f_1(x)$ to be also a uniform function of x and then conclude again that $c_k = d_k$, unless $\rho_1 \rho_2 = 0$. Suppose that $\rho_1 \rho_2 = 0$. Then not both ρ_1 and ρ_2 vanish, for we have assumed $\rho_1 \neq \rho_2$. Suppose $\rho_2 = 0$ and $\rho_1 \neq 0$. Then ϑ_1 reduces to Y_1, ϑ_3 retains its form, but ϑ_2 loses its meaning. Instead of (14), we consider the system

$$\vartheta_3 = \frac{Y_2'}{Y_2} - \frac{\rho_1'}{\rho_1} \log Y_2 = f_3(x), \quad \vartheta_1 = f_1(x),$$

from which the validity of the theorem which we are attempting to prove follows easily for this special case.

In every case, then, Y_1, Y_2 are functions uniform except for $x = a_1, \ldots a_m$, while by a circuit around a_i, (Y_1, Y_2) are transformed into $e^{\rho_1 t_k} Y_1$, $e^{\rho_2 t_k} Y_2$, t_k being a constant depending upon the coefficients of $f_2(x)$ and $f_3(x)$, namely, being equal to $c_k = d_k$, which can, moreover, be chosen at will. *Thus it is seen that functions y_1, y_2 exist which undergo arbitrary substitutions A_i of our group, when x makes a circuit around arbitrarily assigned points a_i in the plane.* In general, the point $x = \infty$ will also occur as branch point with the substitution A_{m+1}, so that $A_1 A_2 \ldots A_m A_{m+1} = 1$. If, however, $A_1, \ldots A_m$ are chosen, so that $A_1 \ldots A_m = 1$, y_1, y_2 will be uniform at infinity. It is interesting to note that if ρ_1, ρ_2, f_2, f_3 are rational functions of x, $f_1(x)$ is necessarily a transcendental function or a constant.

Our group is algebraic if $\frac{\rho_2}{\rho_1}$ is a rational number, for then ϑ_1 is algebraic. This is essentially a theorem of Maurer's.

If we put $\lambda_1 = -\lambda_2 = i$, $\rho_1 = -\rho_2$, we get the group of rotations with the invariant $y_1^2 + y_2^2$. If the condition $\rho_1 = -\rho_2$ alone is upheld, we obtain a group with a homogeneous quadratic invariant.

§2.—*One-parameter Groups. Case II.*

Now let $\rho_1 = \rho_2 = \rho$ and $\lambda_1 = \lambda_2 = \lambda$, so that

$$D = (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 0, \tag{1}$$

and hence

$$\rho = \tfrac{1}{2}(\psi_{11} + \psi_{22}), \quad \lambda = -\frac{\psi_{22} - \psi_{11}}{2\psi_{21}} = +\frac{2\psi_{12}}{\psi_{22} - \psi_{11}}. \tag{2}$$

The finite equations of the group are

$$\left.\begin{aligned}\eta_1 &= (1+\psi_{21}\lambda t)\,e^{\rho t}y_1 - \lambda^2\psi_{21}\,te^{\rho t}y_2,\\ \eta_2 &= \psi_{21}\,te^{\rho t}y_1 + e^{\rho t}(1-\lambda\psi_{21}t)\,y_2,\end{aligned}\right\} \tag{3}$$

or in a more symmetrical but less convenient form:

$$\left.\begin{aligned}\eta_1 &= [1-\tfrac{1}{2}(\psi_{22}-\psi_{11})\,t]\,e^{\rho t}y_1 + \tfrac{1}{2}\lambda(\psi_{22}-\psi_{11})\,te^{\rho t}y_2,\\ \eta_2 &= -\frac{\psi_{22}-\psi_{11}}{2\lambda}\,te^{\rho t}y_1 + [1+\tfrac{1}{2}(\psi_{22}-\psi_{11})\,t]\,e^{\rho t}y_2.\end{aligned}\right\} \tag{3a}$$

The invariants are easily found. We have from (3)

$$(\eta_1-\lambda\eta_2) = e^{\rho t}(y_1-\lambda y_2) \tag{4}$$

and
$$\frac{\eta_2}{\eta_1-\lambda\eta_2} = \frac{\psi_{21}ty_1+(1-\lambda\psi_{21}t)\,y_2}{y_1-\lambda y_2} = \frac{\psi_{21}t\,(y_1-\lambda y_2)+y_2}{y_1-\lambda y_2},$$

or
$$\frac{\eta_2}{\eta_1-\lambda\eta_2} = \frac{y_2}{y_1-\lambda y_2} + \psi_{21}t. \tag{5}$$

Thus the expression

$$\vartheta_1 = (y_1-\lambda y_2)^{\psi_{21}}\,e^{-\rho\frac{y_2}{y_1-\lambda y_2}} \tag{6}$$

is an absolute invariant of our group, which might also have been found by integration.

The differential invariants of the first order can easily be formed from (4) and (5). They are

$$\left.\begin{aligned}\vartheta_2 &= \frac{d}{dx}\left[\frac{1}{\rho}\log(y_1-\lambda y_2)\right],\\ \vartheta_3 &= \frac{d}{dx}\left[\frac{1}{\psi_{21}}\,\frac{y_2}{y_1-\lambda y_2}\right],\end{aligned}\right\} \tag{7}$$

and the relation between ϑ_2, ϑ_3 and $\frac{d\vartheta_1}{dx}$ is

$$\frac{d}{dx}\,\frac{\log\vartheta_1}{\rho\psi_{21}} = \vartheta_2-\vartheta_3, \tag{8}$$

or
$$\frac{1}{\vartheta_1}\,\frac{d\vartheta_1}{dx} = \frac{1}{\rho\psi_{21}}\,\frac{d(\rho\psi_{21})}{dx}\log\vartheta_1 - \rho\psi_{21}(\vartheta_2-\vartheta_3). \tag{8a}$$

These expressions are not valid when $\rho=0$ or $\psi_{21}=0$. ρ and ψ_{21} cannot both vanish, for then the group would contain only the identical transformation. If $\rho=0$, $\psi_{21}\neq 0$, we have the independent invariants

$$y_1-\lambda y_2,\quad y_1'-\lambda y_2'-\lambda' y_2,\quad \frac{d}{dx}\left(\frac{1}{\psi_{21}}\,\frac{y_2}{y_1-\lambda y_2}\right), \tag{9}$$

and if $\psi_{21}=0$, $\rho\neq 0$,

$$\frac{y_2}{y_1-\lambda y_2},\quad \frac{d}{dx}\left(\frac{y_2}{y_1-\lambda y_2}\right),\quad \frac{d}{dx}\left(\frac{1}{\rho}\log\left(y_1-\lambda y_2\right)\right).^* \tag{9a}$$

In the general case, consider the system of differential equations

$$\vartheta_2=f_2(x),\quad \vartheta_3=f_3(x),\quad \vartheta_1=f(x),$$

where $f_i(x)$ are uniform functions of x, and

$$\log f(x)=\rho\psi_{21}\int\left(f_2(x)-f_3(x)\right)dx.$$

Applying our previous argument, word for word, we find that functions y_1, y_2 are thus determined, uniform, finite and continuous everywhere except for points $a_1, \ldots . a_m$ arbitrarily chosen. Moreover, these points are branch-points of such a nature, that after a circuit of x around a_i, y_1, y_2 will have undergone an arbitrary linearoid substitution of our group. The special cases offer no exceptions.

It will be noticed that in both Case I and Case II the numerable subgroup generated by $A_1, \ldots . A_m$ is such, that all of its substitutions are transformed into the canonical form of linear substitutions by the same transformation.

§3.—*Two-parameter Groups.*

Let the infinitesimal transformations be

$$\left.\begin{aligned} U_1f&=(\phi_{11}y_1+\phi_{12}y_2)\,q_1+(\phi_{21}y_1+\phi_{22}y_2)\,q_2=\xi_1q_1+\xi_2q_2,\\ U_2f&=(\psi_{11}y_1+\psi_{12}y_2)\,q_1+(\psi_{21}y_1+\psi_{22}y_2)\,q_2=\eta_1q_1+\eta_2q_2,\end{aligned}\right\} \tag{1}$$

where

$$q_i=\frac{\partial f}{\partial y_i},\qquad (i=1,\,2)$$

---

* The case $\lambda=\infty$, in which y_2 is an invariant, causes no difficulty.

and where ϕ_{ik} and ψ_{ik} are uniform functions of x such that $U_1 f$ and $U_2 f$ generate a two-parameter linearoid group.

According to Lie's general theory, U_1, U_2 generate a group if, and only if,

$$(U_1,\, U_2) = c_1 U_1 + c_2 U_2, \tag{2}$$

where c_1 and c_2 are constants. Equation (2) is equivalent to the system of relations

$$\left.\begin{aligned} A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} &&= c_1\phi_{11} + c_2\psi_{11}, \\ A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} &&= c_1\phi_{12} + c_2\psi_{12}, \\ A_3 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} &&= c_1\phi_{21} + c_2\psi_{21}, \\ A_4 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} &&= c_1\phi_{22} + c_2\psi_{22}. \end{aligned}\right\} \tag{3}$$

Between the left members of these equations, the following relations hold:

$$\left.\begin{aligned} \phi_{21}A_2 + \phi_{12}A_3 + (\phi_{11} - \phi_{22})A_1 &= 0, \\ \psi_{21}A_2 + \psi_{12}A_3 + (\psi_{11} - \psi_{22})A_1 &= 0, \\ A_4 + \qquad\qquad\qquad A_1 &= 0. \end{aligned}\right\} \tag{4}$$

According to a general method of Lie, we consider the following cases in order:

I. $(U_1,\, U_2) = 0$, and no equation of the form $X_1 U_1 f + X_2 U_2 f = 0$, where X_1, X_2 are functions of x, y_1, y_2.

II. $(U_1,\, U_2) = 0, \qquad X_1 U_1 f + X_2 U_2 f = 0.$

In cases I and II, $c_1 = c_2 = 0$. If either of these is not zero, we can always put $c_1 = 1$, $c_2 = 0$. Thus, we have two more cases:

III. $(U_1,\, U_2) = U_1, \qquad X_1 U_1 f + X_2 U_2 f \neq 0.$

IV. $(U_1,\, U_2) = U_1, \qquad X_1 U_1 f + X_2 U_2 f = 0.$

We proceed to consider these cases in detail.

Case I.

According as certain of the quantities ϕ_{ik} and ψ_{ik} vanish, the infinitesimal transformations may assume different forms, which we shall write down, although they may belong to the same *type* in Lie's terminology. We thus obtain not only all *types* but all *forms* of linearoid and linear groups.

First, suppose that neither ϕ_{21} nor ψ_{21} vanish identically. Then, since $A_1 = 0$, either $\phi_{12} = \psi_{12} = 0$, or neither ϕ_{12} nor ψ_{12} vanish. But the case $\phi_{12} = \psi_{12} = 0$ is identical except, as to notation, with $\phi_{21} = \psi_{21} = 0$, which will be discussed later. In Case I^a, then, take ϕ_{21}, ψ_{21}, ϕ_{12}, ψ_{12} all different from zero. Equations $A_i = 0$ reduce to $A_1 = 0$, $A_3 = 0$, or

$$\frac{\phi_{12}}{\phi_{21}} = \frac{\psi_{12}}{\psi_{21}} = \omega_1(x), \quad \frac{\phi_{11} - \phi_{22}}{\phi_{21}} = \frac{\psi_{11} - \psi_{22}}{\psi_{21}} = \omega_2(x),$$

and, therefore, if we write

$$\phi_{22} = \phi_1, \quad \phi_{21} = \phi_1', \quad \psi_{22} = \phi_2, \quad \psi_{21} = \phi_2',$$

we have for form I^a,

$$\text{I}^a. \quad U_i f = [(\phi_i + \omega_2 \phi_i') y_1 + \omega_1 \phi_i' y_2] q_1 + [\phi_i' y_1 + \phi_i y_2] q_2. \quad (i = 1, 2).$$

Moreover, in order that the group may really be linearoid, i. e., in order that the finite equations of the group may have uniform coefficients, the characteristic equation of the general infinitesimal transformation must be reducible, i. e.,

$$\omega_2^2 + 4\omega_1 = \omega^2,$$

where ω is a uniform function of x. This condition is necessary and sufficient.

For form I^b, we have $\psi_{21} = 0$, $\phi_{21} \neq 0$,

$$\text{I}^b. \quad \begin{cases} U_1 f = (\phi_{11} y_1 + \phi_{12} y_2) q_1 + (\phi_{21} y_1 + \phi_{22} y_2) q_2, \\ U_2 f = \qquad \psi(x) (y_1 q_1 + y_2 q_2), \end{cases}$$

and the group is linearoid if the one-parameter group $U_1 f$ is, i. e., if

$$D = (\phi_{11} + \phi_{22})^2 - 4(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) = \omega(x)^2,$$

where $\omega(x)$ is a uniform function of x.

In Case I^c, we have $\psi_{21} = 0$, $\phi_{21} = 0$. Only one equation, $A_i = 0$, is left, viz., $A_2 = 0$, or

$$(\psi_{11} - \psi_{22}) \phi_{12} = (\phi_{11} - \phi_{22}) \psi_{12}.$$

In Case I^{c1}, let $\psi_{12}=0$, but $\phi_{12}\neq 0$. Then $\psi_{11}=\psi_{22}$ and we have U_1, U_2 of the same form as I^b, only that $\phi_{21}=0$. If ϕ_{12} and ψ_{12} are both different from zero, we have

$$\frac{\phi_{11}-\phi_{22}}{\phi_{12}}=\frac{\psi_{11}-\psi_{22}}{\psi_{12}}=\omega,$$

and, therefore,

$$I^{c2}.\quad \begin{cases} U_1 f=[(\phi_{22}+\omega\phi_{12})\,y_1+\phi_{12}\,y_2]\,q_1+\phi_{22}\,y_2\,q_2,\\ U_2 f=[(\psi_{22}+\omega\psi_{12})\,y_1+\psi_{12}\,y_2]\,q_1+\psi_{22}\,y_2\,q_2.\end{cases}$$

If $\phi_{12}=\psi_{12}=0$, we have

$$I^{c3}.\quad \begin{cases} U_1 f=\phi_{11}\,y_1\,q_1+\phi_{22}\,y_2\,q_2,\\ U_2 f=\psi_{11}\,y_1\,q_1+\psi_{22}\,y_2\,q_2.\end{cases}$$

Both groups are linearoid without further restrictions.

In all of these cases it is assumed that

$$\Delta=\xi_1\eta_2-\xi_2\eta_1\neq 0,$$

for otherwise there would be an equation of the form

$$X_1\,U_1 f+X_2\,U_2 f=0.$$

Now, according to a general theorem of Lie's, if $U_1 f$ and $U_2 f$ generate a two-parameter group, Δ is a relative invariant. Let us put

$$(U_1,\ U_2)=c\,U_1,\quad c=1 \text{ or } 0,$$

to include both cases. Then we find, employing a general method of Lie's (Transformationsgruppen, vol. I, p. 242),

$$U_1\,\Delta=(\phi_{11}+\phi_{22})\,\Delta,\quad U_2\,\Delta=(\psi_{11}+\psi_{22}-c)\,\Delta. \tag{5}$$

In our particular case, $c=0$, and hence

$$U_1\,\Delta=(\phi_{11}+\phi_{22})\,\Delta,\quad U_2\,\Delta=(\psi_{11}+\psi_{22})\,\Delta. \tag{6}$$

We can always reduce the groups of Case I to the form

$$\frac{\partial f}{\partial \mathfrak{y}_1},\ \frac{\partial f}{\partial \mathfrak{y}_2},$$

i. e., to a group of two translations, by a transformation of coordinates. For, in order that this may be possible, it is necessary and sufficient to be able to choose new variables $\mathfrak{y}_1$ and $\mathfrak{y}_2$, so that for any function $f(y_1, y_2)$,

$$\xi_1 \frac{\partial f}{\partial y_1} + \xi_2 \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial \mathfrak{y}_1}, \quad \eta_1 \frac{\partial f}{\partial y_1} + \eta_2 \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial \mathfrak{y}_2},$$

or, what amounts to the same thing, if we put $f = \mathfrak{y}_1$ and $f = \mathfrak{y}_2$ respectively, so that

$$\left\{\begin{array}{ll} \xi_1 \dfrac{\partial \mathfrak{y}_1}{\partial y_1} + \xi_2 \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = 1, & \xi_1 \dfrac{\partial \mathfrak{y}_2}{\partial y_1} + \xi_2 \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = 0, \\ \eta_1 \dfrac{\partial \mathfrak{y}_1}{\partial y_1} + \eta_2 \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = 0, & \eta_1 \dfrac{\partial \mathfrak{y}_2}{\partial y_1} + \eta_2 \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = 1, \end{array}\right.$$

or that

$$\left.\begin{array}{ll} \dfrac{\partial \mathfrak{y}_1}{\partial y_1} = + \dfrac{\eta_2}{\Delta}, & \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = - \dfrac{\eta_1}{\Delta}, \\ \dfrac{\partial \mathfrak{y}_2}{\partial y_1} = - \dfrac{\xi_2}{\Delta}, & \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = + \dfrac{\xi_1}{\Delta}, \end{array}\right\} \tag{7}$$

Now, these equations can be integrated. For equations (6) are verified, and they are precisely the conditions of integrability for (7), and $\Delta \neq 0$. Or, what means the same thing, our groups are such as to make

$$d\mathfrak{y}_1 = \frac{\eta_2 \, dy_1 - \eta_1 \, dy_2}{\Delta}, \quad d\mathfrak{y}_2 = \frac{-\xi_2 \, dy_1 + \xi_1 \, dy_2}{\Delta} \tag{7a}$$

complete differentials, so that $\mathfrak{y}_1$ and $\mathfrak{y}_2$ are obtained by integrating (7a).

We thus obtain the canonical variables in every case. We find in Case I^a

$$\left.\begin{array}{l} \Delta = (\phi_1 \phi_2' - \phi_2 \phi_1')(y_1^2 - \omega_1 y_2^2 - \omega_2 y_1 y_2) = (\phi_1 \phi_2' - \phi_2 \phi_1')(y_1 - \alpha y_2)(y_1 - \beta y_2), \\ \mathfrak{y}_1 = \quad \tfrac{1}{2} \phi_2' \log \Delta + \dfrac{\tfrac{1}{2} \phi_2' \omega_2 - \phi_2}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{1}{\beta - \alpha} \log \dfrac{y_1 - \beta y_2}{y_1 - \alpha y_2}, \\ \mathfrak{y}_2 = -\tfrac{1}{2} \phi_1' \log \Delta - \dfrac{\tfrac{1}{2} \phi_1' \omega_2 - \phi_1}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{1}{\beta - \alpha} \log \dfrac{y_1 - \beta y_2}{y_1 - \alpha y_2}, \end{array}\right\} \tag{8}$$

if $\alpha \neq \beta$. If $\alpha = \beta = \frac{1}{2} \omega_2$, we obtain

$$\left.\begin{array}{l} \mathfrak{y}_1 = \quad \tfrac{1}{2} \phi_2' \log \Delta - \dfrac{\tfrac{1}{2} \phi_2' \omega_2 - \phi_2}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{y_2}{y_1 - \tfrac{1}{2} \omega_2 y_2}, \\ \mathfrak{y}_2 = - \tfrac{1}{2} \phi_1' \log \Delta + \dfrac{\tfrac{1}{2} \phi_1' \omega_2 - \phi_1}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{y_2}{y_1 - \tfrac{1}{2} \omega_2 y_2}. \end{array}\right\} \tag{8a}$$

In Case I^{b}, we have

$$\Delta = \psi(x)\,[-\phi_{21}y_1^2 + \phi_{12}y_2^2 + (\phi_{11}-\phi_{22})\,y_1y_2] = \psi(x)(y_1-\alpha y_2)(y_1-\beta y_2),$$

and if $\alpha \neq \beta$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= -\frac{1}{\phi_{21}}\,\frac{1}{\beta-\alpha}\log\frac{y_1-\beta y_2}{y_1-\alpha y_2}, \\ \mathfrak{y}_2 &= -\frac{1}{2\psi(x)}\log\Delta + \frac{\phi_{11}-\phi_{22}}{2\psi(x)\,\phi_{21}}\log\frac{y_1-\beta y_2}{y_1-\alpha y_2}. \end{aligned}\right\} \tag{9}$$

If $\alpha = \beta = -\frac{1}{2}\,\dfrac{\phi_{11}-\phi_{22}}{\phi_{21}}$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{1}{\phi_{21}}\,\frac{y_2}{y_1-\alpha y_2}, \\ \mathfrak{y}_2 &= -\frac{1}{2\psi(x)}\log\Delta - \frac{\phi_{11}-\phi_{22}}{2\psi(x)\,\phi_{21}}\,\frac{y_2}{y_1-\alpha y_2}. \end{aligned}\right\} \tag{9a}$$

Group I^{c1} has the same form as I^{b}, but (9) and (9a) are useless in this case since $\phi_{21} = 0$. We find for I^{c1}, if $\phi_{11}-\phi_{22} \neq 0$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= -\frac{1}{\phi_{11}-\phi_{22}}\log\left(\frac{y_1}{y_2}+\frac{\phi_{12}}{\phi_{11}-\phi_{22}}\right), \\ \mathfrak{y}_2 &= -\frac{1}{2\psi}\log\Delta - \frac{1}{2\psi}\log\left(\frac{y_1}{y_2}+\frac{\phi_{12}}{\phi_{11}-\phi_{22}}\right), \end{aligned}\right\} \tag{10}$$

and if $\phi_{11}-\phi_{22} = 0$,

$$\mathfrak{y}_1 = -\frac{1}{\phi_{12}}\,\frac{y_1}{y_2}, \quad \mathfrak{y}_2 = -\frac{1}{2\psi}\log(\phi_{12}\,\psi y_2^2). \tag{10a}$$

In Case I^{c2}, we find

$$\Delta = (\phi_{12}\psi_{22} - \phi_{22}\psi_{12})\,y_2(\omega y_1 + y_2).$$

If $\omega \neq 0$, we have

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{\psi_{22}}{\omega(\phi_{12}\psi_{22}-\phi_{22}\psi_{12})}\log\left(\frac{y_1}{y_2}+\frac{1}{\omega}\right) - \frac{\psi_{12}}{\phi_{12}\psi_{22}-\phi_{22}\psi_{12}}\log y_2, \\ \mathfrak{y}_2 &= -\frac{\phi_{22}}{\omega(\phi_{12}\psi_{22}-\phi_{22}\psi_{12})}\log\left(\frac{y_1}{y_2}+\frac{1}{\omega}\right) + \frac{\phi_{12}}{\phi_{12}\psi_{22}-\phi_{22}\psi_{12}}\log y_2, \end{aligned}\right\} \tag{11}$$

and if $\omega = 0$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{1}{\phi_{12}\psi_{22} - \phi_{22}\psi_{12}}\left(\ \ \psi_{22}\frac{y_1}{y_2} - \psi_{12}\log y_2\right), \\ \mathfrak{y}_2 &= \frac{1}{\phi_{12}\psi_{22} - \phi_{22}\psi_{12}}\left(-\phi_{22}\frac{y_1}{y_2} + \phi_{12}\log y_2\right). \end{aligned}\right\} \tag{11a}$$

In case I^{c3}, we have

$$\left.\begin{aligned} \Delta &= (\phi_{11}\psi_{22} - \phi_{22}\psi_{11})\, y_1 y_2, \\ \mathfrak{y}_1 &= \frac{1}{\phi_{11}\psi_{22} - \phi_{22}\psi_{11}}\ (\psi_{22}\log y_1 - \psi_{11}\log y_2), \\ \mathfrak{y}_2 &= \frac{1}{\phi_{11}\psi_{22} - \phi_{22}\psi_{11}}\,(-\phi_{22}\log y_1 + \phi_{11}\log y_2). \end{aligned}\right\} \tag{12}$$

If the groups are linearoid, in all cases where the determinant Δ has been separated into its factors $y_1 - \alpha y_2$ and $y_1 - \beta y_2$, α and β are uniform functions of x.

In every case $\frac{d\mathfrak{y}_1}{dx}$ and $\frac{d\mathfrak{y}_n}{dx}$ are differential invariants of the first order. The differential equations

$$\frac{d\mathfrak{y}_i}{dx} = r_i(x) \qquad (i = 1,\ 2) \tag{13}$$

are linearoid, belonging to the given two-parameter group. It is easy to see that $r_i(x)$ may be chosen as rational functions of x in such a way that the functions y_1, y_2, defined as solutions of (13), are uniform everywhere except in the vicinity of certain arbitrarily assigned points $a_1 \ldots . a_m$, and that when x makes a circuit around a_i, y_1, y_2 undergo an arbitrary substitution A_i of the group.

These differential equations can also be obtained in another way. We have seen that Δ is a relative invariant. Factor Δ. It is found that each factor of Δ is also a relative invariant, which shows that the logarithm of each factor verifies a non-homogeneous linear differential equation. But this method fails if Δ is a perfect square, since it furnishes only one differential equation in that case.

We see again that the numerable subgroup generated by $A_1, \ldots . A_m$ is such that all of its substitutions are reduced to the canonical form by the same transformation.

Case II.

In Case II, we have $(U_1, U_2)=0$, and besides a relation $X_1U_1f+X_2U_2f=0$, so that $\Delta=0$. We shall show that in this case $X_1:X_2$ is a function of x alone. Generally, since

$$X_1U_1f+X_2U_2f=0,$$

we can put

$$U_1=\rho_1 U, \quad U_2=\rho_2 U,$$

where U is an infinitesimal transformation, which we will call of degree λ if its coefficients contain y_1, y_2 in the λ^{th} power. Now, either U is of degree 1, and ρ_1 and ρ_2 are of degree 0, or U is of degree zero, and ρ_1 and ρ_2 of degree 1. In the first case we have at once $(U_1, U_2)=0$, $\dfrac{X_1}{X_2}=$ function of x. In the second case, let

$$Uf=\lambda q_1+\mu q_2, \quad \rho_1=\phi_1 y_1+\phi_2 y_2, \quad \rho_2=\psi_1 y_1+\psi_2 y_2.$$

Then we find

$$(U_1, U_2)=(\phi_1\psi_2-\phi_2\psi_1)(\mu y_1-\lambda y_2)\,Uf.$$

In order that this may be zero, we must have $\phi_1\psi_2-\phi_2\psi_1=0$, i. e., $\rho_1=\rho \times$ function of $x=\rho_2\,.\,\rho(x)$ or $U_1f=\rho_2(x)\,U_2f$.

We have, therefore, shown that, in Case II, the infinitesimal transformations always have the form

$$\text{II.} \quad U_1f, \quad X(x)\,U_1f.$$

We can also, at this point, set down the form for Case IV. In that case, $(U_1, U_2)=U_1$, and only the second case, viz., that U is of degree zero, and $\rho_1\rho_2$ of degree 1, can occur. The condition $(U_1, U_2)=U_1$ gives

$$\mu=\frac{\phi_1}{\phi_1\psi_2-\phi_2\psi_1}, \quad \lambda=-\frac{\phi_2}{\phi_1\psi_2-\phi_2\psi_1},$$

and hence

$$\text{IV.} \quad \begin{cases} U_1f=\dfrac{\phi_1 y_1+\phi_2 y_2}{\phi_1\psi_2-\phi_2\psi_1}(-\phi_2 q_1+\phi_1 q_2), \\[2ex] U_2f=\dfrac{\psi_1 y_1+\psi_2 y_2}{\phi_1\psi_2-\phi_2\psi_1}(-\phi_2 q_1+\phi_1 q_2). \end{cases}$$

Let us return to Case II. Suppose, first, that the characteristic equation of $U_1 f$ has unequal roots. Put, as in §1,

$$\mathfrak{y}_1 = Y_1^{\rho_1} Y_2^{-\rho_2}, \quad \mathfrak{y}_2 = \frac{1}{\rho_2} \log Y_1,$$

these being the canonical variables for the one-parameter group generated by $U_1 f$. Then $\mathfrak{y}_1$ is an invariant for the two-parameter group, and $\mathfrak{y}_2$ is transformed into

$$\bar{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_1 + c_2 X$$

by the most general transformation of the two-parameter group, c_1, c_2 being arbitrary constants. We obtain, therefore, the following differential invariant of the second order:

$$\frac{d}{dx} \frac{\frac{d\mathfrak{y}_2}{dx}}{\frac{dX}{dx}}.$$

The typical system of linearoid differential equations for this group is, therefore,

$$\frac{d}{dx} \frac{\frac{d\mathfrak{y}_2}{dx}}{\frac{dX}{dx}} = r(x); \quad \mathfrak{y}_1 = Y_1^{\rho_1} Y_2^{-\rho_2} = s(x),$$

If $\rho_1 = \rho_2$, we must put for $\mathfrak{y}_1$ and $\mathfrak{y}_2$ the expressions of §2.

Let $a_1, \ldots . a_m$ be the poles of $r(x)$. Let a represent any one of them, and suppose that in the vicinity of $x = a$, we have

$$r(x) = \ldots . + \frac{c_2}{(x-a)^2} + \frac{c_1}{x-a} + c_0 + \ldots .$$

Then

$$\frac{d\mathfrak{y}_2}{dx} = \frac{dX}{dx} \Big[\ldots . - \frac{c_2}{x-a} c_1 \log(x-a) + c_0' + c_0 (x-a) + \ldots . \Big]$$

$$= c_1 X' \log(x-a) + \ldots . + \frac{e_1}{x-a} + e_0 + \ldots . ,$$

since $X(x)$ is also a uniform function of x. Moreover, c_1 and e_1 can be regarded as absolutely arbitrary. Integration gives

$$\mathfrak{y}_2 = e_1 \log(x-a) + c_1 \left[X \log(x-a) - \int X \frac{dx}{x-a}\right] + P(x-a) + P_1\left(\frac{1}{x-a}\right),$$

or if

$$\frac{X}{x-a} = \cdots \frac{l_1}{x-a} + l_0 + \cdots,$$

$$\mathfrak{y}_2 = (e_1 - c_1 l_1) \log(x-a) + c_1 X \log(x-a) + Q(x-a) + Q_1\left(\frac{1}{x-a}\right).$$

This shows at once that e_1 and c_1 can be chosen so that $\mathfrak{y}_2$ shall undergo an arbitrary substitution of the form

$$\overline{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_1 + Xc_2,$$

when x makes a circuit around a. The proof holds for any number of points a_k by resolving $r(x)$ into partial fractions. Correspondingly, y_1, y_2 undergo arbitrary linearoid substitutions of our group when x makes circuits about the arbitrarily selected points a_i.

Case III.

This case is characterized by the relations

$$(U_1, U_2) = U_1, \quad X_1 U_1 f + X_2 U_2 f \neq 0,$$

so that $c_1 = 1$, $c_2 = 0$.

Regarding equations (3) of this paragraph as equations for determining ϕ_{ik} as functions of ψ_{ik}, we can write them as follows:

$$\left.\begin{array}{lllll}
-\phi_{11} & -\psi_{21}\phi_{12} & +\psi_{12}\phi_{21} & & = 0, \\
-\psi_{12}\phi_{11} & +(\psi_{11}-\psi_{22}-1)\phi_{12} & & +\psi_{12}\phi_{22} & = 0, \\
+\psi_{21}\phi_{11} & & +(\psi_{22}-\psi_{11}-1)\phi_{21} & -\psi_{21}\phi_{22} & = 0, \\
 & +\psi_{21}\phi_{12} & -\psi_{12}\phi_{21} & -\phi_{22} & = 0.
\end{array}\right\} \qquad (14)$$

The determinant of these equations, which is

$$1 - (\psi_{11} - \psi_{22})^2 - 4\psi_{21}\psi_{12},$$

must vanish, i. e., we must have

$$D_2 = (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 1, \qquad (15)$$

i. e., since D_2 is the discriminant of the characteristic equation of the infinitesimal transformation $U_2 f$, the roots of this equation must be distinct, and their difference must be equal to unity.

We obtain, further, by adding the first and last of equations (14).

$$\phi_{11} + \phi_{22} = 0. \tag{16}$$

Further, we notice that from the first relation (4), using equations (3), putting $c_1 = 1$, $c_2 = 0$, we obtain

$$2\phi_{12}\phi_{21} + (\phi_{11} - \phi_{22})\phi_{11} = 0,$$

or, according to (16),

$$\phi_{12}\phi_{21} + \phi_{11}^2 = 0. \tag{17}$$

But the discriminant of the characteristic equation of $U_1 f$ is

$$D_1 = (\phi_{11} - \phi_{22})^2 + 4\phi_{12}\phi_{21} = 4\left[\phi_{11}^2 + \phi_{12}\phi_{21}\right],$$

and, therefore,

$$D_1 = 0,$$

i. e., the roots of that equation are equal. But more than that, they are zero, for this equation, or

$$\omega^2 - (\phi_{11} + \phi_{22})\,\omega + \phi_{11}\phi_{22} - \phi_{12}\phi_{21} = 0$$

reduces to $\omega^2 = 0$ if (16) and (17) are fulfilled.

The following theorem is, therefore, true: *If $U_1 f$, $U_2 f$ generate a linearoid two-parameter group, such that*

$$(U_1, U_2) = U_1,$$

then the characteristic equation of $U_1 f$ has both of its roots equal to zero, while the roots of the characteristic equation of $U_2 f$ are distinct, and the difference between them is unity.

The first part of this theorem, if applied to linear groups, is a special case of a general theorem of Killing's. The theorem can obviously be generalized.

From (14) we find

$$\phi_{12} = -\frac{2\psi_{12}\phi_{11}}{1 - (\psi_{11} - \psi_{22})}, \quad \phi_{21} = \frac{2\psi_{21}\phi_{11}}{1 + \psi_{11} - \psi_{22}}, \quad \phi_{22} = -\phi_{11},$$

if $1 \pm (\psi_{11} - \psi_{22}) \neq 0$. Call this Case III^{a}. Since ϕ_{11} is still arbitrary, put it equal to $[1 - (\psi_{11} - \psi_{22})^2]\,\phi(x)$, where $\phi(x)$ is also an arbitrary function of x. We have then

$$\text{III}^{a}.\quad \left\{\begin{array}{l} U_1 f = \phi(x)\,[\{1 - (\psi_{11} - \psi_{22})^2\}\,y_1 - 2\psi_{12}\,\{1 + (\psi_{11} - \psi_{22})\}\,y_2]\,q_1 \\ \qquad + \phi(x)\,[2\psi_{21}\,\{1 - (\psi_{11} - \psi_{22})\}\,y_1 - \{1 - (\psi_{11} - \psi_{22})^2\}\,y_2]\,q_2, \\ U_2 f = (\psi_{11} y_1 + \psi_{12} y_2)\,q_1 + (\psi_{21} y_1 + \psi_{22} y_2)\,q_2 \\ \qquad\qquad\qquad\qquad (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 1. \end{array}\right.$$

Moreover, we find

$$\Delta = (\xi_1 \eta_2 - \xi_2 \eta_1) = \phi\psi_{21}\,(1 - \psi_{11} - \psi_{22})(1 - \psi_{11} + \psi_{22})[y_1 - \lambda y_2]^2,$$
$$\lambda = \frac{1 + \psi_{11} - \psi_{22}}{2\psi_{21}} = \frac{2\psi_{12}}{1 - (\psi_{11} - \psi_{22})}.$$

Since we must have in our case $\Delta \neq 0$, the case that

$$\psi_{11} + \psi_{22} = 1$$

is also excluded from III^{a}.

Now let $1 - (\psi_{11} - \psi_{22}) = 0$, $1 + \psi_{11} - \psi_{22} = 2$. Equations (14) in this case (III^{b}) reduce to

$$\left\{\begin{array}{rrrrrl} -\phi_{11} & & -\psi_{21}\phi_{12} & +\psi_{12}\phi_{21} & & = 0, \\ -\psi_{12}\phi_{11} & & & & +\psi_{12}\phi_{22} & = 0, \\ +\psi_{12}\phi_{11} & & & -2\phi_{21} & -\psi_{21}\phi_{22} & = 0, \\ & & +\psi_{21}\phi_{12} & -\psi_{12}\phi_{21} & -\phi_{22} & = 0, \end{array}\right.$$
$$D_2 - 1 = 4\psi_{12}\psi_{21} = 0.$$

We have subcases to consider. First, in Case III^{b1}, let $\psi_{12} \neq 0$. Then $\psi_{21} = 0$, and since generally we find from the above equations

$$\psi_{12}(\phi_{11} - \phi_{22}) = 0, \quad \phi_{11} + \phi_{22} = 0,$$

we have $\phi_{11} = \phi_{22} = 0$, and from the first equation also $\phi_{21} = 0$, while $\phi_{12} = \phi$ remains arbitrary. We have then

$$\text{III}^{b1}.\quad \left\{\begin{array}{l} U_1 f = \phi(x)\,y_2 q_1, \\ U_2 f = (\psi_{11} y_1 + \psi_{12} y_2)\,q_1 + (\psi_{11} - 1)\,y_2 q_2. \end{array}\right.$$

Moreover,

$$\Delta = \phi\,(\psi_{11} - 1)\,y_2^2,$$

so that ψ_{11} must be different from unity.

If $\psi_{12}=0$ and $\psi_{21}=0$, we need only put $\psi_{12}=0$ in III^{b1}. Let $\psi_{12}=0$, $\psi_{21}\neq 0$. Then

$$\phi_{12}=-\frac{\phi_{11}}{\psi_{21}}, \quad \phi_{21}=\psi_{21}\phi_{11}, \quad \phi_{22}=-\phi_{11}.$$

Thus we obtain, putting $\phi_{11}=\phi$,

$$\mathrm{III}^{b2}. \quad \begin{cases} U_1 f=\phi(x)\left(y_1-\dfrac{1}{\psi_{21}}y_2\right)[q_1+\psi_{21}q_2], \\ U_2 f=\psi_{11}y_1q_1+[\psi_{21}y_1+(\psi_{11}-1)y_2]q_2, \\ \quad \Delta=\phi\dfrac{1-\psi_{11}}{\psi_{21}}(\psi_{21}y_1-y_2)^2, \end{cases}$$

the case $\psi_{11}=1$ being again excluded.

Now let $1+\psi_{11}-\psi_{22}=0$, $1-(\psi_{11}-\psi_{22})=2$. We find if $\psi_{12}\neq 0$,

$$\mathrm{III}^{c1}. \quad \begin{cases} U_1 f= \quad \phi(x)(y_1-\psi_{12}y_2)[q_1+\dfrac{1}{\psi_{12}}q_2], \\ U_2 f= \quad (\psi_{11}y_1+\psi_{12}y_2)q_1+(1+\psi_{11})y_2q_2, \\ \quad \Delta=-\dfrac{\psi_{11}}{\psi_{12}}(y_1-\psi_{12}y_2)^2, \end{cases}$$

and since $\Delta\neq 0$, we must have $\psi_{11}\neq 0$.

If $\psi_{12}=0$, we have

$$\mathrm{III}^{c2}. \quad \begin{cases} U_1 f= \quad \phi(x)y_1q_2, \\ U_2 f= \quad \psi_{11}y_1q_1+[\psi_{21}y_1+(1+\psi_{11})y_2]q_2, \\ \quad \Delta=-\phi(x)\psi_{11}y_1^2. \end{cases}$$

We have now obtained all of the different forms of groups belonging to Case III. In every case Δ is a relative invariant. Since Δ is a perfect square, we have thus a linear invariant for every form belonging to Case III.

By a transformation of coordinates, the groups of Case III may be reduced to the form

$$\frac{\partial f}{\partial \mathfrak{y}_2}, \quad \mathfrak{y}_1\frac{\partial f}{\partial \mathfrak{y}_1}+\mathfrak{y}_2\frac{\partial f}{\partial \mathfrak{y}_2}, \tag{19}$$

as may be shown just as in Lie's "Vorlesungen über Differentialgleichungen," pp. 421–422. The following is a direct proof of this, and furnishes, besides, the formulæ for the transformation.

In order that U_1f, U_2f may assume the form (19), it is necessary and sufficient that $\mathfrak{y}_1$ and $\mathfrak{y}_2$ be determined, so that

$$U_1(\mathfrak{y}_1) = 0\,, \quad U_2(\mathfrak{y}_1) = \mathfrak{y}_1\,, \quad U_1(\mathfrak{y}_2) = 1\,, \quad U_2(\mathfrak{y}_2) = \mathfrak{y}_2\,. \tag{20}$$

Now we have §3, (5),

$$U_1(\Delta) = 0\,, \quad U_2(\Delta) = (\psi_{11} + \psi_{22} - 1)\,\Delta\,, \tag{21}$$

so that we may put

$$\mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}. \tag{22}$$

We know further, taking the case that $\phi_{21} \neq 0$, that

$$U_1\left(\frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2}\right) = 1\,, \quad \lambda = -\,\frac{\phi_{22} - \phi_{11}}{2\phi_{21}} = +\,\frac{2\phi_{12}}{\phi_{22} - \phi_{11}}\,, \tag{23}$$

for the roots of the characteristic equation of U_1f are equal. Therefore,

$$U_1\left(\mathfrak{y}_2 - \frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2}\right) = 0\,,$$

and therefore, since every solution of $U_1f = 0$ is a function of x and Δ,

$$\mathfrak{y}_2 = \frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2} + \Phi(\Delta, x)\,. \tag{24}$$

The further condition $U_2(\mathfrak{y}_2) = \mathfrak{y}_2$ easily furnishes the most general form possible for $\Phi(\Delta, x)$, which contains an arbitrary function of x. But this is unnecessary, for we easily find

$$U_2\left(\frac{y_2}{y_1 - \lambda y_2} + \psi_{21}\right) = U_2\left(\frac{y_2}{y_1 - \lambda y_2}\right) = \frac{y_2}{y_1 - \lambda y_2} + \psi_{21}\,.$$

We can, therefore, put

$$\left.\begin{aligned} \mathfrak{y}_1 &= \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}\,, \\ \mathfrak{y}_2 &= \frac{1}{\phi_{21}}\left[\frac{y_2}{y_1 - \lambda y_2} + \psi_{21}\right]. \end{aligned}\right\} \tag{25}$$

Thus we find in the different cases the following sets of canonical variables:

$$\text{III}^{a}.\quad \begin{cases} \mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}, \\ \mathfrak{y}_2 = \dfrac{1}{2(1-\psi_{11}+\psi_{22})}\left[\dfrac{1}{\psi_{21}}\;\dfrac{y_2}{y_1-\lambda y_2}+1\right], \\ \lambda = \dfrac{1+\psi_{11}-\psi_{22}}{2\psi_{21}} = \dfrac{2\psi_{12}}{1-(\psi_{11}-\psi_{22})}; \end{cases}$$

$$\text{III}^{b1}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2(\psi_{11}-1)}}, \quad \mathfrak{y}_2 = \frac{\psi_{12}}{\phi}\left(\frac{1}{\psi_{12}}\;\frac{y_1}{y_2}+1\right);$$

$$\text{III}^{b2}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2(\psi_{11}-1)}}, \quad \mathfrak{y}_2 = \frac{1}{\phi}\left(\frac{1}{\psi_{21}}\;\frac{y_2}{y_1-\frac{1}{\psi_{21}}y_2}+1\right);$$

$$\text{III}^{c1}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2\psi_{11}}}, \quad \mathfrak{y}_2 = \frac{\psi_{12}}{\phi}\;\frac{y_2}{y_1-\psi_{12}y_2};$$

$$\text{III}^{c2}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2\psi_{11}}}, \quad \mathfrak{y}_2 = \frac{\psi_{21}}{\phi}\left(\frac{1}{\psi_{21}}\;\frac{y_2}{y_1}+1\right).$$

All of the forms III are truly linearoid. For if we compute the discriminant of the characteristic equation belonging to the general infinitesimal transformation of the group

$$c_1 U_1 f + c_2 U_2 f,$$

we find it to be equal to c_2^2, so that the difference of the roots is c_2. This completes a former theorem, so that we now know that *if U_1, U_2 generated a two-parameter group for which $(U_1, U_2) = U_1$, then the characteristic equation belonging to $U_1 f$ has its roots both zero, that belonging to $U_2 f$ has its roots distinct and their difference equal to* 1, *and that belonging to $c_1 U_1 + c_2 U_2$ has its roots distinct if $c_2 \neq 0$, and their difference equal to c_2.*

This theorem can also be read as a purely algebraic one, about the roots of three equations,

$$|\phi_{ik} - \delta_{ik}\rho| = 0, \quad |\psi_{ik} - \delta_{ik}\rho| = 0, \quad |c_1\phi_{ik} + c_2\psi_{ik} - \delta_{ik}\rho| = 0,$$
$$(\delta_{ik} = 0, \quad i \neq k, \quad \delta_{ii} = 1),$$

if the relations (3) of this section for $c_1 = 1$, $c_2 = 0$ take place between ϕ_{ik} and ψ_{ik}, and is doubtless true generally.

The two-parameter group induced by $U_1 f$, $U_2 f$ on $\mathfrak{y}_1$, $\mathfrak{y}_2$ is

$$\overline{\mathfrak{y}}_1 = a\mathfrak{y}_1, \quad \overline{\mathfrak{y}}_2 = a\mathfrak{y}_2 + b, \tag{26}$$

so that we have the following independent differential invariants of the first order

$$\frac{1}{\mathfrak{y}_1}\,\frac{d\mathfrak{y}_1}{dx}, \quad \frac{1}{\mathfrak{y}_1}\,\frac{d\mathfrak{y}_2}{dx}. \tag{27}$$

It may further be shown that to every transformation of the form (26) corresponds a transformation of our group in y_1 and y_2, which is perfectly definite, if only the meaning of such expressions as

$$a^{\frac{\psi_{11}+\psi_{22}-1}{2}} = e^{\frac{\psi_{11}+\psi_{22}-1}{2}\log a}$$

is made clear, by choosing one among the infinity of values of $\log a$. If, then, we can show that functions $\mathfrak{y}_1$, $\mathfrak{y}_2$ exist which undergo arbitrary substitutions of the form (26) for circuits of x around arbitrary points a_k, it follows that y_1, y_2 undergo the corresponding substitutions of our group for the same circuits. If, instead of $\mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}$, we take $\mathfrak{y}_1 = (y_1 - \lambda y_2)^{\frac{2}{\psi_{11}+\psi_{22}-1}}$, we can also avoid the introduction of such algebraic branch points as are obtained, for instance, by solving the equations for Case III^a, which contain the square root

$$\sqrt{\phi\,(1-\psi_{11}+\psi_{22})(1-\psi-\psi_{22})\,\psi_{21}}.$$

Everything, then, depends upon the proof that functions $\mathfrak{y}_1$, $\mathfrak{y}_2$ exist, which for circuits of x around a_k, undergo the substitutions

$$\overline{\mathfrak{y}}_1 = \alpha_k\,\mathfrak{y}_1, \quad \overline{\mathfrak{y}}_2 = \alpha_k\,\mathfrak{y}_2 + \beta_k, \qquad (k = 1, 2, \ldots . m).$$

If there are such functions, $r_1(x)$ and $r_2(x)$, in

$$\frac{1}{\mathfrak{y}_1}\,\frac{d\mathfrak{y}_1}{dx} = r_1(x), \quad \frac{1}{\mathfrak{y}_1}\,\frac{d\mathfrak{y}_2}{dx} = r_2(x),$$

will be uniform functions. If we have

$$r_1(x) = \frac{1}{2\pi i}\sum_{k=1}^{m}\log \alpha_k\,\frac{1}{(x-a_k)} + \ldots . ,$$

we obtain

$$\mathfrak{y}_1 = \prod_{k=1}^{m}(x-a_k)^{\frac{1}{2\pi i}\log \alpha_k}\,u(x),$$

where $u(x)$ is a uniform function, and $\mathfrak{y}_1$ has the required property.

We then have

$$\mathfrak{y}_2 = \int_{\lambda_0}^{x} \prod_{k=1}^{m} (x - a_k)^{\frac{1}{2\pi i}\log \alpha_k} v(x)\, dx,$$

where $v(x)$ is also a uniform function of x, and where, for convenience, we have taken a definite integral.

Assume that $v(x)$ has no zeros or poles coincident with any point a_k, and that the expansion of $v(x)$ in the vicinity of its poles b contains no term of the form $(x - b)^{-1}$. Then the value of $\mathfrak{y}_2$ is changed only by circuits around a_k. For such a circuit we have

$$\bar{\mathfrak{y}}_2 = \alpha_k \mathfrak{y}_2 + (1 - \alpha_k) \int_{x_0}^{a_k} \prod_{k=1}^{m} (x - a_k)^{\frac{1}{2\pi i}\log \alpha_k} v(x)\, dx,$$

provided that the definite integral on the right member is well defined, or an equivalent equation involving a loop integral if this is not the case. If we take

$$v(x) = c_0 + c_1 x + \dots + c_{m-1} x^{m-1},$$

$c_0, \dots c_{m-1}$ can be so chosen that we obtain

$$\bar{\mathfrak{y}}_2 = \alpha_k \bar{\mathfrak{y}}_2 + \beta_k,$$

α_k and β_k being arbitrary constants, with this limitation, however, that if $\alpha_k = 1$, $\beta_k = 0$, so *that if a_k is no branch point for $\mathfrak{y}_1$, it is not for $\mathfrak{y}_2$.*

Case IV.

$$(U_1, U_2) = U_1, \quad X_1 U_1 + X_2 U_2 = 0.$$

We have already found for this case

$$\begin{cases} U_1 f = \dfrac{\phi_1 y_1 + \phi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1} (-\phi_2 q_1 + \phi_1 q_2) = \rho_1 U, \\ U_2 f = \dfrac{\psi_1 y_1 + \psi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1} (-\phi_2 q_1 + \phi_1 q_2) = \rho_2 U, \end{cases}$$

where

$$\phi_1 \psi_2 - \phi_2 \psi_1 \neq 0.$$

We have the absolute invariant

$$\mathfrak{y}_1 = \phi_1 y_1 + \phi_2 y_2 = \rho_1,$$

and find further

$$U_1\left(\frac{\rho_2}{\rho_1}\right)=1, \quad U_2\left(\frac{\rho_2}{\rho_1}\right)=\frac{\rho_2}{\rho_1}.$$

Putting, therefore,

$$\mathfrak{y}_1=\phi_1 y_1+\phi_2 y_2, \quad \mathfrak{y}_2=\frac{\psi_1 y_1+\psi_2 y_2}{\phi_1 y_1+\phi_2 y_2},$$

these variables are transformed by the two-parameter group

$$\bar{\mathfrak{y}}_1=\mathfrak{y}_1, \quad \bar{\mathfrak{y}}_2=\alpha\mathfrak{y}_2+\beta.$$

Our differential invariant is, therefore,

$$\frac{d\log}{dx}\frac{d\mathfrak{y}_2}{dx}.$$

The further discussion is similar to Case III.

§4.—*Three-parameter Groups.*

The first possible composition of a three-parameter group, assuming the group to be simple, is

$$I \quad (U_1, U_2)=U_1, \quad (U_1, U_3)=2U_2, \quad (U_2, U_3)=U_3. \tag{1}$$

These relations give at once

$$\left.\begin{aligned}&\phi_{11}+\phi_{22}=0, \quad \psi_{11}+\psi_{22}=0, \quad \chi_{11}+\chi_{22}=0,\\ &D_1=(\phi_{11}-\phi_{22})^2+4\phi_{12}\phi_{21}=D_3=(\chi_{11}-\chi_{22})^2+4\chi_{12}\chi_{21}=0,\\ &D_2=(\psi_{11}-\psi_{22})^2+4\psi_{12}\psi_{21}=1,\end{aligned}\right\} \tag{2}$$

where ϕ_{ik}, ψ_{ik}, χ_{ik} are the coefficients of U_1f, U_2f, U_3f respectively. By the methods of §3 we find

$$\left.\begin{aligned}&\phi_{11}=\phi, \quad \phi_{12}=-\frac{2\psi_{12}}{1-2\psi_{11}}\phi, \quad \phi_{21}=\frac{2\psi_{21}}{1+2\psi_{11}}\phi, \quad \phi_{22}=-\phi,\\ &\chi_{11}=\chi, \quad \chi_{12}=\frac{2\psi_{12}}{1+2\psi_{11}}\chi, \quad \chi_{21}=-\frac{2\psi_{21}}{1-2\psi_{11}}\chi, \quad \chi_{22}=-\chi,\end{aligned}\right\} \tag{3}$$

where ϕ and χ are arbitrary functions of x. Equations (3) are the consequences of $(U_1, U_2)=U_1$, and $(U_2, U_3)=U_3$, provided that $\psi_{11}\neq\pm\frac{1}{2}$. Now, from

$(U_1, U_3) = 2U_2$, we obtain

$$\left.\begin{aligned} \phi_{21}\chi_{12} - \phi_{12}\chi_{21} &= 2\psi_{11}, \\ \phi_{12}\chi_{11} - \phi_{11}\chi_{12} + \phi_{22}\chi_{12} - \phi_{12}\chi_{22} &= 2\psi_{12}, \\ \phi_{11}\chi_{21} - \phi_{21}\chi_{11} + \phi_{21}\chi_{22} - \phi_{22}\chi_{21} &= 2\psi_{21}, \\ \phi_{12}\chi_{21} - \phi_{21}\chi_{12} &= 2\psi_{22}. \end{aligned}\right\} \tag{4}$$

Substituting the expressions (3), we obtain

$$-4\phi\chi\frac{\psi_{11}}{1-4\psi_{11}^2} = \psi_{11} = -\psi_{22},$$

$$\psi_{12} = -4\psi_{12}\frac{\phi\chi}{1-4\psi_{11}^2} = -\frac{\phi\chi}{\psi_{21}}, \quad \psi_{21} = -4\psi_{21}\frac{\phi\chi}{1-4\psi_{11}^2} = -\frac{\phi\chi}{\psi_{12}}.$$

or

$$\psi_{12}\psi_{21} = -\phi\chi. \tag{5}$$

as the only additional condition.

The roots of the characteristic equation belonging to $U_2 f$ are $+\frac{1}{2}$ and $-\frac{1}{2}$. *The roots of the characteristic equation belonging to the general infinitesimal transformation of the group,* $c_1 U_1 + c_2 U_2 + c_3$, *are found to be*

$$\rho_1 = -\rho_2 = \tfrac{1}{2}(c_2^2 - 2c_1 c_3). \tag{6}$$

The group is similar (ähnlich) to the special linear group by the transformation

$$\eta_1 = \frac{2}{1+2\psi_{11}}\left(y_1 - \frac{2\psi_{12}}{1-2\psi_{11}}y_2\right), \quad \eta_2 = y_1 + \frac{2\psi_{12}}{1+2\psi_{11}}y_2. \tag{7}$$

For we find

$$U_1(\eta_1) = 0, \quad U_1(\eta_2) = \eta_1; \quad U_3(\eta_1) = -\eta_2, \quad U_3(\eta_2) = 0;$$
$$U_2(\eta_1) = -\tfrac{1}{2}\eta_1, \quad U_2(\eta_2) = +\tfrac{1}{2}\eta_2,$$

so that if we introduce η_1, η_2 as new variables, the infinitesimal transformations become

$$\eta_1\bar{q}_2, \quad \tfrac{1}{2}(-\eta_1\bar{q}_1 + \eta_2\bar{q}_2), \quad -\eta_2\bar{q}_1,$$

which generate the special linear group.

The same is true, as is easily verified, if $\psi_{11} = \pm\frac{1}{2}$, the case not included in the above investigation.

The next possible composition is

$$(U_1, U_2) = 0, \quad (U_1, U_3) = U_1, \quad (U_2, U_3) = cU_2, \qquad (c \neq 0, \neq 1) \tag{8}$$

A rather long but not uninteresting discussion shows that there are no linearoid groups of this composition.

Next we may have (Case II):

$$(U_1, U_2) = 0, \quad (U_1, U_3) = U_1, \quad (U_2, U_3) = U_2. \tag{9}$$

We have at once

$$\left.\begin{aligned} U_1 f &= \phi\left[\left(y_1 - \frac{2\chi_{12}}{1-(\chi_{11}-\chi_{22})}\, y_2\right) q_1 + \left(\frac{2\chi_{21}}{1+\chi_{11}-\chi_{22}}\, y_1 - y_2\right) q_2\right], \\ U_2 f &= \frac{\psi}{\phi}\, U_1 f, \quad U_3 f = (\chi_{11} y_1 + \chi_{12} y_2)\, q_1 + (\chi_{21} y_1 + \chi_{22} y_2)\, q_2 \\ &\qquad\qquad (\chi_{11}-\chi_{22})^2 + 4\chi_{12}\chi_{21} = 1 \end{aligned}\right\} \tag{10}$$

if $\chi_{11} - \chi_{22} \neq \pm 1$. If $\chi_{11} - \chi_{22} = \pm 1$, $U_1 f$ and $U_2 f$ change into forms such as as $\mathrm{III}^{b1} \ldots\ldots \mathrm{III}^{c2}$ of §3, both being always of the same form, so that $U_1 f$ and $U_2 f$ have in this case always the same path-curves.

There is no group of the composition

$$(U_1, U_2) = 0, \quad (U_1, U_3) = U_1, \quad (U_2, U_3) = U_1 + U_2.$$

The next possibility is

$$(U_1, U_2) = 0, \quad (U_1, U_3) = U_1, \quad (U_2, U_3) = 0, \tag{11}$$

which gives

$$\left.\begin{aligned} U_1 f &= \phi\left[\left(y_1 - \frac{2\chi_{12}}{1-(\chi_{11}-\chi_{22})}\, y_2\right) q_1 + \left(\frac{2\chi_{21}}{1+\chi_{11}-\chi_{22}}\, y_1 - y_2\right) q_2\right], \\ U_2 f &= \psi\,(y_1 q_1 + y_2 q_2), \\ U_3 f &= (\chi_{11} y_1 + \chi_{12} y_2)\, q_1 + (\chi_{21} y_1 + \chi_{22} y_2)\, q_2, \end{aligned}\right\} \tag{12}$$

where

$$(\chi_{11}-\chi_{22})^2 + 4\chi_{12}\chi_{21} = 1, \text{ and } \chi_{11} - \chi_{22} \neq \pm 1. \tag{12a}$$

If $\chi_{11} - \chi_{22} = \pm 1$, $U_1 f$ assumes one of the forms $\mathrm{III}^{b1} \ldots\ldots \mathrm{III}^{c3}$ of §3, but $U_2 f$ remains the same as in (12).

Next we may have

$$(U_1, U_2) = 0, \quad (U_1, U_3) = 0, \quad (U_2, U_3) = U_1, \tag{13}$$

from which we find either

$$\left.\begin{aligned} U_1 f &= \phi\,(y_1 q_1 - y_2 q_2), \\ U_2 f &= \psi\,(y_1 q_1 + y_2 q_2), \quad U_3 f = \chi\,(y_1 q_1 + y_2 q_2), \end{aligned}\right\} \tag{14}$$

or

$$\left.\begin{aligned} U_1 f &= \chi_{21}(\psi_{11} - \psi_{22})\, q_2 = \psi_{21}(\chi_{11} - \chi_{22})\, q_2, \\ U_2 f &= \psi_{11}\, y_1 q_1 + (\psi_{21}\, y_1 + \psi_{22}\, y_2)\, q_2, \quad \psi_{22} \neq 0, \\ U_3 f &= \chi_{11}\, y_1 q_1 + \left[\chi_{21}\, y_1 + \left(\chi_{11} - \frac{\phi_{21}}{\psi_{21}}\right) y_2\right] q_2, \end{aligned}\right\} \tag{14a}$$

or a similar group obtained by taking $\psi_{12} \neq 0$, $\psi_{21} = 0$.

There are, finally, a number of cases corresponding to

$$(U_1, U_2) = 0, \quad (U_1, U_3) = 0, \quad (U_2, U_3) = 0, \tag{15}$$

which it is necessary to write down.

The differential equations belonging to the several groups are easily found. In the first case the linearoid system is the transformed of a linear system of the form

$$\eta_i'' + p_1 \eta_i' + p_2 \eta_i = 0, \qquad (i = 1, 2), \tag{16}$$

by the equations (7), the transformation group of (16), in the sense of Picard, being the special linear homogeneous group.

In the next case, if we take for $\mathfrak{y}_1$, $\mathfrak{y}_2$ the canonical variables of Case III, §3, we have

$$U_1(\mathfrak{y}_1) = 0, \quad U_3(\mathfrak{y}_1) = \mathfrak{y}_1; \quad U_1(\mathfrak{y}_2) = 1, \quad U_3(\mathfrak{y}_2) = \mathfrak{y}_2,$$

and, therefore,

$$U_2(\mathfrak{y}_1) = 0, \quad U_2(\mathfrak{y}_2) = \lambda(x) = \frac{\psi}{\phi}.$$

The three-parameter group induced upon $\mathfrak{y}_1$, $\mathfrak{y}_2$ is, therefore,

$$\overline{\mathfrak{y}_1} = \alpha \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \alpha \mathfrak{y}_2 + \beta \lambda(x) + \gamma,$$

so that

$$\frac{1}{\mathfrak{y}_1} \frac{d\mathfrak{y}_1}{dx} = r_1(x), \quad \lambda'' \frac{\mathfrak{y}_2'}{\mathfrak{y}_1} - \lambda \frac{\mathfrak{y}_2''}{\mathfrak{y}_1} = r_2(x),$$

are the differential equations, invariant under this group, λ', λ'', etc., denoting the first and second derivatives.

The function-theoretic nature of the functions $\mathfrak{y}_1$ and $\mathfrak{y}_2$ is easily investigated. As in §3; if a_k $(k = 1, 2, \ldots . m)$ are the poles of $r_1(x)$, and $u(x)$ denotes a uniform function of x,

$$\mathfrak{y}_1 = \prod_{k=1}^{m} (x - a_k)^{r_k} u(x), \quad r_k = \frac{1}{2\pi i} \log \alpha_k,$$

so that, for a circuit around a_k, $\mathfrak{y}_1$ is multiplied by α_k.

Then

$$\mathfrak{y}_2' = \frac{\lambda'}{\lambda_0'}\left[c - \lambda_0'\int_{x_0}^{x} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\,dx\right],$$

$$\mathfrak{y}_2 = c' + \int_{x_0}^{x} \frac{\lambda'}{\lambda_0'}\left[c - \lambda_0'\int_{x_0}^{x} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\,dx\right]dx,$$

where $x = x_0$ is an ordinary point, and λ_0' denotes the value of λ' for $x = x_0$. c and c' are constants of integration.

Thus, for a circuit around a_k, we notice that $\mathfrak{y}_2'$ changes into

$$\overline{\mathfrak{y}_2'} = \alpha_k \mathfrak{y}_2' + \frac{(1-\alpha_k)}{\lambda_0'}\left[c - \lambda_0'\int_{x_0}^{a_k} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\,dx\right]\lambda'(x),$$

where, if $\alpha_k \neq 1$, the factor of $\lambda'(x)$ may be made to assume arbitrarily assigned values β_k by proper choice of the parameters still arbitrary in $u(x)$ and $r_2(x)$. And $\mathfrak{y}_2$ itself changes into

$$\overline{\mathfrak{y}_2} = \alpha_k \mathfrak{y}_2 + \beta_k \lambda(x) + \gamma_k,$$

where

$$\gamma_k = (1-\alpha_k)\int_{x_0}^{a_k} \mathfrak{y}_2'\,dx - \beta_k \lambda(a_k) + \beta^k \lambda(x_0),$$

by a proper choice of the infinitely many parameters still remaining in $u(x)$ and $r_2(x)$, can also be made to assume arbitrarily assigned values. This does not take into account possible exceptions, nor does it inform us as to the simplest functions of this kind. It suffices for our present purpose to note that, in general, functions exist with the arbitrary branch points a_k belonging to the substitutions

$$\overline{\mathfrak{y}_1} = \alpha_k \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \alpha_k \mathfrak{y}_2 + \beta_k \lambda(x) + \gamma_k,$$

where $\lambda(x)$ is a uniform function, and correspondingly there are functions y_1, y_2 with the same branch points and the corresponding linearoid substitutions.

Similar results are obtained from the groups of form (12), (14) and (14a). For groups of composition (15) suppose, first, that $U_i f\ (i = 1, 2, 3)$ have the same path-curves. Then, as we have seen in §3, Case II, $U_i f = \phi_i(x)\, Uf$. Calling $\mathfrak{y}_1, \mathfrak{y}_2$ the canonical variables of the group Uf, these quantities are transformed into

$$\overline{\mathfrak{y}_1} = \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \mathfrak{y}_2 + c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x),$$

so that $\mathfrak{y}_2$ verifies a non-homogeneous linear differential equation of the third order, the corresponding homogeneous equation having ϕ_1, ϕ_2, ϕ_3 as a fundamental system.

If the path-curves of $U_i f$ are not all the same, suppose that $U_1 f$ and $U_2 f$ have distinct path-curves. Then $U_1 f$ and $U_2 f$ can be reduced, as in §3, to the canonical forms $\frac{\partial f}{\partial \mathfrak{y}_1}$, $\frac{\partial f}{\partial \mathfrak{y}_2}$, which gives for $U_3 f$ the form

$$\phi(x) \frac{\partial f}{\partial \mathfrak{y}_1} + \psi(x) \frac{\partial f}{\partial \mathfrak{y}_2},$$

ϕ and ψ being functions of x only. The group of $\mathfrak{y}_1 \mathfrak{y}_2$ is then

$$\overline{\mathfrak{y}}_1 = \mathfrak{y}_1 + c_1 + c_3 \phi(x), \quad \overline{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_2 + c_3 \psi(x),$$

so that both $\mathfrak{y}_1$ and $\mathfrak{y}_2$ verify non-homogeneous linear differential equations.

§5.—*Four and more Parameter Groups.*

Let $U_1 f \dots . U_r f$ generate an r parameter linearoid group. If, in these infinitesimal transformations, we put $x = a$, the group becomes linear, and has $m \leqq 4$ parameters. Let $V_1 f, \dots . V_r f$ be what becomes of $U_1 f, \dots . U_r f$ when x is put equal to a.

Now, according to Lie's classification of binary linear groups, and as can be easily deduced from our results also, such a linear group is either the general linear group, or the special linear group, or, finally, a group conjugate under the general linear group with a group of the form

$$\eta_1 = \alpha y_1, \quad \eta_2 = \beta y_1 + \gamma y_2, \tag{1}$$

where, moreover, α, β, γ will depend upon a, if (1) represents the group generated by $V_1, \dots . V_r$, and may have less than three essential parameters. But then the group generated by $U_1, \dots . U_r$ will be of the same form with x in place of a. Such groups we will leave aside, as they do not present any peculiarity essentially different from those already studied.

We shall, therefore, suppose that $V_1, \dots . V_r$ generate either the general or the special linear homogeneous group. In either case there will be three infinitesimal transformations, say V_1, V_2, V_3 which generate the special linear group among $V_1, \dots . V_r$. Then U_1, U_2, U_3 must generate a three-parameter group with the same composition as the special linear group, which, moreover, we can suppose to be that group itself, only a linearoid transformation being necessary for that purpose, as shown in §4. Thus we shall have

$$U_1 = y_1 q_1, \quad U_2 = \tfrac{1}{2}(-y_1 q_1 + y_2 q_2), \quad U_3 = -y_2 q_1,$$

and obviously all other linearoid infinitesimal transformations can be expressed in the form

$$U_k = \phi_1^{(k)} U_1 + \phi_2^{(k)} U_2 + \phi_3^{(k)} U_3 + \phi_4^{(k)} (y_1 q_1 + y_2 q_2), \quad (k = 4, 5, \ldots . r).$$

But (U_1, U_k) must be expressed as a linear function of $U_1, \ldots . U_r$ with constant coefficients. This requires $\phi_1^{(k)}, \phi_2^{(k)}, \phi_3^{(k)}$ to be constants. But $c_1 U_1 + c_2 U_2 + c_3 U_3$ already occurs in the group. Thus all other infinitesimal transformations of the group can be written in the form

$$U_k f = \phi_k(x)(y_1 q_1 + y_2 q_2), \quad (k = 4, 5, \ldots . r), \tag{2}$$

so that

$$(U_1, U_2) = U_1, \quad (U_1, U_3) = 2U_2, \quad (U_2, U_3) = U_3,$$
$$(U_1, U_k) = (U_2, U_k) = (U_3, U_k) = 0, \qquad (k = 4, 5, \ldots . r).$$

The finite equations of this group are

$$\eta_1 = \rho(\alpha y_1 + \beta y_2), \quad \eta_2 = \rho(\gamma y_1 + \delta y_2), \tag{3}$$

where

$$\rho = e^{\sum_{k=4}^{r} c_k \phi_k(x)}, \tag{4}$$

and the determinant

$$\alpha\delta - \beta\gamma = 1.$$

This latter restriction, however, is inessential, as $\phi_4(x)$ may, in particular, be equal to unity.

The corresponding differential equations are found as follows: The special linear group, i. e., the invariant subgroup U_1, U^2, U_3 has the differential invariants

$$u = y_1 y_2' - y_2 y_1', \quad v = y_1 y_2'' - y_2 y_1'', \quad w = y_1' y_2'' - y_2' y_1'',$$

and these are, therefore, transformed by $U_4, \ldots . U_r$ by an $r - 3$ parameter group, which we shall now investigate. We have, denoting by $U_k'' f$ the twice extended operators $U_k f$,

$$U_k'' f = \phi_k(y_1 q_1 + y_2 q_2) + (\phi_k y_1' + \phi_k' y_1) q_1' + (\phi_k y_2' + \phi_k' y_2) q_2'$$
$$+ (\phi_k y_1'' + 2\phi_k' y_1' + \phi_k'' y_1) q_1'' + (\phi_k y_2'' + 2\phi_k' y_2' + \phi_k'' y_2) q_2'',$$

where

$$q_i^{(k)} = \frac{\partial f}{\partial y_i^{(k)}}, \qquad (i, k = 1, 2).$$

We find

$$\left.\begin{aligned} U_k''(u) &= \phantom{-\phi_k'' u + {}} 2\phi_k u, \\ U_k''(v) &= \phantom{-\phi_k'' u + {}} 2\phi_k' u + 2\phi_k v, \\ U_k'(w) &= -\phi_k'' u + \phi_k' v + 2\phi_k w, \end{aligned}\right\} \tag{5}$$

so that u, v, w are transformed by an r-3 parameter group whose finite equations are

$$\left.\begin{aligned} \overline{u} &= e^{2\phi} u, \\ \overline{v} &= e^{2\phi} [2\phi' u + v], \\ \overline{w} &= e^{2\phi} [(\phi'^2 - \phi'') u + \phi' v + w], \end{aligned}\right\} \tag{5a}$$

where $\phi = c_4 \phi_4 + \ldots\ldots + c_r \phi_r, \ldots\ldots \phi'' = c_4 \phi_4'' + \ldots\ldots + c_r \phi_r'';$ (5b)

the group is a ternary linearoid group.

We have, therefore,

$$\left(\overline{\frac{v}{u}}\right) = \frac{v}{u} + 2\phi',$$

$$\left(\overline{\frac{w}{u}}\right) = \frac{w}{u} + \phi' \frac{v}{u} + \phi'^2 - \phi'',$$

whence

$$\left(\overline{\frac{w}{u} - \tfrac{1}{4} \frac{v^2}{u^2}}\right) = \frac{w}{u} - \tfrac{1}{4} \frac{v^2}{u^2} - \phi'',$$

so that

$$\frac{w}{u} - \tfrac{1}{4} \frac{v^2}{u^2} + \tfrac{1}{2} \frac{d}{dx} \frac{v}{u} = I \tag{6}$$

is an absolute invariant under the group.

Now $\frac{v}{u}$ obviously verifies a non-homogeneous linear differential equation of order $r - 3$. The corresponding homogeneous equation has the fundamental system $\phi_4', \ldots\ldots \phi_r'$. Let $-\frac{v}{u} = p$, and let

$$\frac{d^{r-3} p}{dx^{r-3}} + s_1 \frac{d^{r-4} p}{dx^{r-4}} + \ldots\ldots + s_{r-3} p = s \tag{7}$$

be the differential equation for p. Let $\frac{w}{u} = q$, then

$$q = I + \tfrac{1}{4} p^2 + \tfrac{1}{2} \frac{dp}{dx}. \tag{8}$$

But, according to the definition of p and q, y_1, y_2 are a fundamental system of the linear differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0. \tag{9}$$

Assuming y_1, y_2 to be uniform excepting multiformities expressible by the equations of our group, $s_1, \dots s_{r-3}, s$, and I are uniform functions of x.

Our linearoid system consists, then, of equations (7), (8) and (9), or, instead of (9), the equivalent system

$$-\frac{y_1y_2'' - y_2y_1''}{y_1y_2' - y_2y_1'} = p, \quad \frac{y_1'y_2'' - y_2'y_1''}{y_1y_2' - y_2y_1'} = q. \tag{9a}$$

This is of the third order, p and q being defined by (7) and (8), the general solutions η_1, η_2 being given in terms of $y_1 y_2$ by the formulæ

$$\eta_1 = \alpha y_1 + \beta y_2, \quad \eta_2 = \gamma y_1 + \delta y_2, \quad \alpha\delta - \beta\gamma = 1.$$

The entire system is, therefore, of order r.

We have incidentally found a new proof for the well-known result that I is an invariant of (9) for the infinite group $\bar{y} = \lambda(x)\, y$.

We have seen that if $y_1 y_2$ are transformed by the group (3), the coefficients p and q of the differential equation (9) which they verify are transformed into

$$\left.\begin{aligned} \bar{p} &= p - 2\sum_{k=4}^{r} c_k \phi_k', \\ \bar{q} &= q + p\sum_{k=4}^{r} c_k \phi_k' + \left(\sum_{k=4}^{r} c_k \phi_k'\right)^2 - \sum_{k=4}^{r} c_k \phi_k''. \end{aligned}\right\} \tag{10}$$

The converse is also true, i. e., if p and q are transformed by (10), y_1, y_2 are transformed by (3). For, let us put

$$y = e^{-\frac{1}{2}\int p\,dx}\mathfrak{y},$$

then (9) becomes

$$\frac{d^2\mathfrak{y}}{dx^2} + I\mathfrak{y} = 0, \tag{11}$$

so that I being an invariant for (10), $\mathfrak{y}_1 \mathfrak{y}_2$ are transformed by a linear transformation with constant coefficients. It has been shown by Klein and Poincaré that I may be chosen as a rational function of x, so that the solutions $\mathfrak{y}_1\, \mathfrak{y}_2$ have the arbitrary branch points $a_1, \ldots . a_m$ and undergo an arbitrary linear substitution A_k when x describes a closed path around a_k, provided that the roots of the fundamental equation belonging to A_k have the absolute value unity.

Moreover, it can be seen from the formula obtained by integrating (7), that s can be determined as rational function of x in such a way that for circuits around a_k, p undergoes arbitrary substitutions of form (10).

We thus have the result that functions y_1, y_2 *can be found with the arbitrary branch points* $a_1, \ldots . a_m$ *and the corresponding arbitrary linearoid substitutions of form* (3), *provided only that the roots of the equations of form*

$$\begin{vmatrix} \alpha - \omega, & \gamma \\ \beta & , \delta - \omega \end{vmatrix} = 0$$

have their modulus equal to one.

These functions are such that the quotient $\frac{y_2}{y_1}$ undergoes projective substitutions with constant coefficients, and, therefore, verifies a differential equation of the form

$$\Delta \left(\begin{matrix} \eta \\ x \end{matrix} \right) = u(x),$$

where $\Delta \left(\begin{matrix} \eta \\ x \end{matrix} \right)$ is the Schwarzian derivative.

§6.—*Conclusion.*

In a former paper,* we have seen that the double-loop integrals which verify the hypergeometric differential equation, generalized by considering α, β, γ as uniform functions of x, are functions uniform except in the vicinity of $x = 0$, 1, ∞, and that if x describes a circuit around one of these points, they undergo a linearoid substitution. These substitutions, we can now say, are not contained in any finite continuous linearoid group. The smallest continuous group containing them is an infinite group of linearoid transformations

$$\eta_i = \phi_{i1}(x)\, y_1 + \phi_{i2}(x)\, y_2, \qquad (i = 1, 2).$$

---

* This Journal, vol. XXI, No. 2.

The differential equations, if any, which these functions verify, cannot, therefore, be found by the methods of this paper. The functions of a similar nature obtained here are all very much simpler, and are, in fact, not essentially new. They are but peculiar combinations, as we now see, of solutions of linear differential equations.

The study of ternary linearoid groups and further generalizations, will be undertaken soon by Mr. F. E. Ross, a graduate student of the University of California.

UNIVERSITY OF CALIFORNIA, BERKELEY, *Oct.* 18, 1899.

A Property of Lines in n-Dimensional Space.

By E. O. Lovett, *Princeton, New Jersey.*

1. The following geometrical facts are well known. A line, i. e., a one-dimensional manifoldness, in the plane has curvature when not every three consecutive points on it are collinear; a line in ordinary space has double curvature when not every four points of it are coplanar; similarly, a line in a flat space of four dimensions is said to have triple curvature when not every five consecutive points of it lie in a Euclidean space of four dimensions; and so on for spaces of any number of dimensions.

If a secant line MM' approach a limiting position as M' approaches M on the curve, the limiting position is called a tangent to the curve at the point M; if a secant plane $MM'M''$ tend to assume a limiting position as M' and M'' approach M, the limiting plane is the osculating plane of the curve at the point M; finally, if the Euclidean space Σ_3, $MM'M''M'''$ tend to a limiting position as $M'M''$ and M''' approach coincidence with M, the limiting Σ_3 is called the osculating Σ_3 of the curve at the point M.

Two consecutive tangents have one point in common; two consecutive osculating planes intersect in a tangent, and three consecutive osculating planes have a common point; two, three or four consecutive osculating Σ_3's have in common, respectively, an osculating plane, a tangent, and a point, of the curve.

Every right line perpendicular to the tangent at M at the point of tangency is by definition a normal to the curve at M; a curve of triple curvature possesses a doubly infinite number of normals at every point; these all lie in a Σ_3 which is called the normal space. Of these ∞^2 normals one only is found in the osculating plane; it is called the principal normal. A simply infinite number of normals are perpendicular to the osculating plane; these are called binormals because each one is perpendicular to two consecutive tangents. One binormal, the principal binormal, is contained in the osculating space, and one, the trinormal, is perpendicular to the osculating space; the latter is so called because it is normal to three consecutive tangents.

The tangent, principal normal, principal binormal, and trinormal at the same point are the principal directions of the curve at this point; they are mutually perpendicular, and each is normal to the Σ_3 formed by the other three. The curve admits of but one osculating plane at a point, but has a simply infinite number of normal planes, all of which are situated in the normal space; one of the normal planes, the principal normal plane, is in the osculating space. The principal normal plane is determined by the principal normal and the principal binormal, since the osculating and normal spaces have the principal normal and principal binormal in common. The normal space is the space perpendicular to the tangent; the space perpendicular to the trinormal is the osculating space; that normal to the principal normal is called the rectifying space; there is no occasion for giving the fourth space a name.

Thus, at every point of the curve there is a tetrarectangular tetraeder whose edges are the four remarkable lines associated with the curve at that point, namely, the tangent, the principal normal, the principal binormal, and the trinormal. It is convenient to assume as origin the movable point M of the curve, the tangent as x-axis, the trinormal as y-axis, the principal binormal as z-axis, and the principal normal as t-axis. Let $\delta\phi$ be the angle between the two consecutive tangents, $\delta\psi$ that between the two adjacent principal binormals, and $\delta\chi$ that between the two adjacent trinormals, when the origin M is shifted along the curve to the neighboring position M'. Then, from the definitions of the principal axes,

$$\cos(t, x') = d\phi, \quad \cos(t, z') = d\psi, \quad \cos(z, y') = d\chi,$$

In order to complete the array of direction-cosines of the axes at M' with respect to those at M, it is only necessary to observe that

$$\cos(x, x') = \cos(y, y') = \cos(z, z') = \cos(t, t') = 1,$$

and that because of the perpendicularity of the axes

$$\cos(x, y') = 0, \quad \cos(x, z') = 0, \quad \cos(x, t') = -d\phi, \; \ldots\ldots;$$

accordingly, the quadrangle of direction-cosines becomes

$$\left.\begin{array}{c|cccc} & x & y & z & t \\ \hline x' & 1 & 0 & 0 & d\phi \\ y' & 0 & 1 & d\chi & 0 \\ z' & 0 & -d\chi & 1 & d\psi \\ t' & -d\phi & 0 & -d\psi & 1 \end{array}\right\} \qquad (1)$$

The ratios of the differentials of ϕ, ψ, χ to ds, i. e., the limits of the ratios of $\delta\phi$, $\delta\psi$, $\delta\chi$ to MM' as M' approaches M, measure the curvatures at M; the first is called the flexion, the second the torsion, and the third the curvature; or perhaps more simply, the first, second and third curvatures. Thus, if

$$ds = \rho d\phi = \tau d\psi = r d\chi,$$

the number ρ, τ, r, the reciprocals of the curvatures, measure three lengths which are called, respectively, the radius of flexion, the radius of torsion, and the radius of curvature. The flexion of a space curve consists of the more or less rapid deviation of the curve from the tangent; the torsion, of the more or less swing from the osculating plane; the curvature, of the rapidity with which the curve leaves the osculating space.

The array (1) shows that in order to discuss the curve in the domain of each of its points, it is sufficient to know the functions ϕ, ψ, χ, i. e., sufficient to give ρ, τ, r as functions of s. Thus, the three equations, the so-called intrinsic equations of the curve

$$f_1(s, \rho, \tau, r) = 0, \quad f_2(s, \rho, \tau, r) = 0, \quad f_3(s, \rho, \tau, r) = 0,$$

determine the curvatures at every point of the curve lying between points at which the tangents may become indeterminate; the equations determine the forms of these arcs but do not fix the positions of the branches in space.

2. The method of the intrinsic analysis used so successfully by Professor E. Cesàro in his "Lezioni di geometria intrinseca" and developed by him for configurations in spaces of any number of dimensions, may be employed to bring to light an interesting property of curves in n-dimensional space.

Let P be a point in space referred to M as origin and generally movable with M; let its coordinates be $x(s)$, $y(s)$, $z(s)$, $t(s)$. Let P' be the point of the trajectory of P corresponding to M'; its coordinates referred to axes at M are $x+\delta x$, $y+\delta y$, $z+\delta z$, $t+\delta t$; its coordinates referred to axes at M' are $x+dx$, $y+dy$, $z+dz$, $t+dt$. Let u, v, w, p be the coordinates of M' referred to M. Then projecting $M'P'$ on the axes of M, we have, by virtue of the array (1),

$$\left.\begin{aligned} x+\delta x &= u+x+dx-(t+dt)\,d\phi,\\ y+\delta y &= v+y+dy-(z+dz)\,d\chi,\\ z+\delta z &= w+z+dz+(y+dy)\,d\chi-(t+dt)\,d\psi,\\ t+\delta t &= p+t+dt+(x+dx)\,d\phi+(z+dz)\,d\psi.\end{aligned}\right\} \tag{2}$$

Confining attention to those curves, for which it is legitimate to assume that the limit of the ratio of the arc to the chord is unity, it follows from the definition of the tangent that

$$\mathcal{L}\frac{u}{k}=1,\quad \mathcal{L}\frac{v}{k}=\mathcal{L}\frac{w}{k}=\mathcal{L}\frac{p}{k}=0,\quad k=\sqrt{u^2+v^2+w^2+p^2},$$

hence

$$\mathcal{L}\frac{u}{\delta s}=1,\quad \mathcal{L}\frac{v}{\delta s}=\mathcal{L}\frac{w}{\delta s}=\mathcal{L}\frac{p}{\delta s}=0,$$

and the formulæ (2) give the fundamental relations of Cesàro's analysis,

$$\left.\begin{aligned} \frac{\delta x}{ds}&=\frac{dx}{ds}-\frac{t}{\rho}+1, & \frac{\delta y}{ds}&=\frac{dy}{ds}-\frac{z}{r},\\ \frac{\delta z}{ds}&=\frac{dz}{ds}+\frac{y}{r}-\frac{t}{\tau}, & \frac{\delta t}{ds}&=\frac{dt}{ds}+\frac{x}{\rho}+\frac{z}{\tau}.\end{aligned}\right\} \tag{3}$$

The latter formulæ give the following as the necessary and sufficient conditions for the immobility of a point, viz.:

$$\frac{dx}{ds}=\frac{t}{\rho}-1,\quad \frac{dy}{ds}=\frac{z}{r},\quad \frac{dz}{ds}=\frac{t}{\tau}-\frac{y}{r},\quad \frac{dt}{ds}=-\frac{x}{\rho}-\frac{z}{\tau}. \tag{4}$$

Now, take the origin of arcs at any point of the curve at which the curvatures, always supposed to be continuous functions of the arc, have finite values; let x, y, z, t be its coordinates with respect to a fundamental tetraeder at a neighboring point. The coordinates x, y, z, t are infinitesimal along with s, and since the conditions (4) ought to be satisfied, we have

$$\mathcal{L}\frac{x}{s}=\mathcal{L}\frac{dx}{ds}=\mathcal{L}\left(\frac{t}{\rho}-1\right)=-1,\quad \mathcal{L}\frac{y}{s}=\mathcal{L}\frac{dy}{ds}=\mathcal{L}\frac{z}{r}=0,$$

$$\mathcal{L}\frac{z}{s}=\mathcal{L}\frac{dz}{ds}=\mathcal{L}\left(\frac{t}{\tau}-\frac{y}{r}\right)=0,\quad \mathcal{L}\frac{t}{s}=\mathcal{L}\frac{dt}{ds}=\mathcal{L}\left(\frac{x}{\rho}+\frac{z}{\tau}\right)=0,$$

then, also,

$$\pounds\,\frac{t}{s^2} = \tfrac{1}{2}\pounds\,\frac{1}{s}\;\frac{dt}{ds} = -\tfrac{1}{2}\pounds\,\frac{1}{s}\left(\frac{x}{\rho} + \frac{z}{\tau}\right) = \frac{1}{2\rho},$$

$$\pounds\,\frac{z}{s^3} = \tfrac{1}{3}\pounds\,\frac{1}{s^2}\left(\frac{t}{\tau} - \frac{y}{r}\right) = \frac{1}{6\rho\tau},$$

$$\pounds\,\frac{y}{s^4} = \tfrac{1}{4}\pounds\,\frac{1}{s^3}\left(\frac{z}{r}\right) = \frac{1}{24\rho\tau r}.$$

The expressions for the coordinates u, v, w, p of M', with regard to the axes of M, are obtained by changing s into $-ds$ in the above formulæ, since transferring the origin of arcs to the point $M'(s+ds)$ infinitely near to $M(s)$ is equivalent to putting $s+ds=0$; accordingly,

$$u = ds, \quad v = \frac{ds^4}{24\rho\tau r}, \quad w = -\frac{ds^3}{6\rho\tau}, \quad p = \frac{ds^2}{\rho}; \tag{5}$$

hence, of all the Σ_3's passing through M, those containing the tangent are characterized by the fact that their distances from points on the curve infinitely near to M are infinitesimals of a higher order, and for one of these, the osculating Σ_3, the distance is infinitesimal of at least the fourth order.

It follows also that every arc, sufficiently small, taken in the neighborhood of the point M is situated wholly on one side of any Σ_3 through the tangent, with the exception of the Σ_3 formed by the tangent, trinormal, and principal binormal. Contrary, then, to the property of curves of double curvature, the osculating space of highest dimensions of a curve of triple curvature lies wholly on one side of the curve. The process by which this result was reached shows that a line of multiple curvature cuts its osculating space of highest dimensions or lies wholly on one side of that space according as the number of dimensions of the space necessary to the existence of the curve is odd or even.

Concerning the Cyclic Subgroups of the Simple Group G of all Linear Fractional Substitutions of Determinant Unity in Two Non-Homogeneous Variables with Coefficients in an Arbitrary Galois Field.[*]

BY LEONARD EUGENE DICKSON.

1. This paper leads to a generalization to the $GF[p^n]$ of certain results due to Professor Burnside[†] upon groups of linear substitutions in the $GF[p]$, i. e., the field of integers taken modulo p, a prime. Numerous variations from his method of treatment have been introduced in the present paper, partly to avoid the separate treatment of the two cases $d = 1$ and $d = 3$, and to enable us to catch in the exceptional cases $p = 2$ and $p = 3$, and partly to minimize the calculations and, on the other hand, to amplify certain proofs left to the reader. Professor Burnside's results for the case $p \equiv 1 \pmod{3}$ are incorrect in two places. The factor 2 should be deleted from $\frac{N}{2(p-1)^2}$ on p. 103 and p. 104 of his paper; in fact, the statements made on p. 103, lines 15–23, are correct, but do not lead to the conclusion stated. A more subtle error was made on p. 102, lines 1–6. There exist three (and not just one) conjugate sets of substitutions of the canonical form there considered. This is made evident at the end of §4 below.

It has been possible to make the present treatment very brief and yet complete as to details, by making continual use of the general results of the paper on the "Canonical Form of a Linear Homogeneous Substitution in a Galois

---

[*] Presented in abstract at the meeting of December 28–29, 1899, of the Chicago Section of the American Mathematical Society.

[†] "On a Class of Groups Defined by Congruences," Proc. Lond. Math. Soc., vol. 26, pp. 58–106.

Field" (Amer. Jour. of Math., vol. XXII, pp. 121–137). We refer to this paper as "the earlier paper."

The distribution into conjugate sets of the substitutions of G is given by the formulæ (6), (7), (8), (9), (10), (11) and (12), the identical substitution forming another set.

From the results of this paper in the special case $p^n = 2^3$, we derive (§13) an immediate proof of the non-isomorphism of the alternating group on eight letters and our group G for $p^n = 2^3$, each being a simple group of order 20160. This result was first established by Miss Schottenfels under the direction of Professor Moore.*

2. The group G of all substitutions of determinant $|\alpha_{ij}| = 1$,

$$S: \quad x' = \frac{\alpha_{11}x + \alpha_{12}y + \alpha_{13}}{\alpha_{31}x + \alpha_{32}y + \alpha_{33}}, \quad y' = \frac{\alpha_{21}x + \alpha_{22}y + \alpha_{23}}{\alpha_{31}x + \alpha_{32}y + \alpha_{33}},$$

in which the coefficients α_{ij} belong to the $GF[p^n]$, is a simple group of order†

$$N \equiv \frac{1}{d}(p^{3n} - 1)(p^{2n} - 1)p^{3n},$$

where d is the greatest common divisor of 3 and $p^n - 1$, so that

$$d = 1, \text{ if } p^n = 3^n \text{ or } 3l - 1; \quad d = 3, \text{ if } p^n = 3l + 1.$$

The equation $\tau^3 = 1$ has in the $GF[p^n]$ a single root $\tau = 1$, if $d = 1$, but has three roots $\theta, \theta^2, \theta^3 \equiv 1$, if $d = 3$. Hence, there are exactly d homogeneous substitutions of determinant unity

$$\xi_i' = \theta^r(\alpha_{i1}\xi_1 + \alpha_{i2}\xi_2 + \alpha_{i3}\xi_3) \qquad (i = 1, 2, 3)$$

which, when taken fractionally, lead to the same non-homogeneous substitution S of determinant unity. The homogeneous and non-homogeneous groups are, therefore, simply isomorphic if $d = 1$. For $d = 3$, we may still work with the homogeneous group in place of G, provided we regard as identical the three substitutions

$$\Sigma, \quad \Theta\Sigma \equiv \Sigma\Theta, \quad \Theta^2\Sigma \equiv \Sigma\Theta^2,$$

where Θ is the homogeneous substitution multiplying each index by θ.

---

* Annals of Mathematics, 2d Ser., vol. I, pp. 147–152.

† Part II of the writer's Chicago dissertation; Annals of Mathematics, vol. XI, 1897, pp. 161–188. Also Burnside, "Theory of Groups," p. 340.

3. We can exhibit G as a permutation-group on $p^{2n}+p^n+1$ letters. Every linear function $A\xi_1+B\xi_2+C\xi_3$, in which A, B, C are marks not all zero of the $GF[p^n]$, can be put into one of the forms

$$\mu(\xi_3+\rho\xi_2+\sigma\xi_1),\quad \mu(\xi_2+\rho\xi_1),\quad \mu\xi_1,$$

where μ, ρ, σ are marks of the $GF[p^n]$ and $\mu\neq 0$. Combining into one system $\{A\xi_1+B\xi_2+C\xi_3\}$ the p^n-1 linear functions $\mu(A\xi_1+B\xi_2+C\xi_3)$, μ denoting in succession the p^n-1 marks $\neq 0$ of the field, we obtain $p^{2n}+p^n+1$ distinct systems,

$$\{\xi_3+\rho\xi_2+\sigma\xi_1\},\quad \{\xi_2+\rho\xi_1\},\quad \{\xi_1\},\qquad [\rho,\sigma \text{ arbitrary marks}].$$

Any ternary homogeneous linear substitution replaces the functions $\mu(A\xi_1+B\xi_2+C\xi_3)$, comprising one system, by linear functions

$$\mu(A\xi_1'+B\xi_2'+C\xi_3')\equiv\mu(\alpha\xi_1+\beta\xi_2+\gamma\xi_3),$$

all belonging to a single system. Hence, it permutes the above $p^{2n}+p^n+1$ symbols amongst themselves. It follows that G is isomorphic with a permutation-group G' on these symbols. But a homogeneous substitution altering none of the symbols must have the form

$$\xi_1'=\alpha\xi_1,\quad \xi_2'=\alpha\xi_2,\quad \xi_3'=\alpha\xi_3.$$

If it have determinant unity, it corresponds in G to the identity. Hence, G is *simply* isomorphic with G'.

The permutation-group G' is doubly transitive. We need only prove that G' contains a permutation converting $\{\xi_1\}$, $\{\xi_2+\xi_1\}$ into respectively

$$\{A\xi_1+B\xi_2+C\xi_3\},\quad \{A'\xi_1+B'\xi_2+C'\xi_3\},$$

the latter being any two distinct symbols, viz.:

$$A:B:C\neq A':B':C'.$$

For the corresponding homogeneous substitution, we may take

$$\xi_1'=A\xi_1+B\xi_2+C\xi_3,\quad \xi_2'=(A'-A)\xi_1+(B'-B)\xi_2+(C'-C)\xi_3,$$
$$\xi_3'=\alpha\xi_1+\beta\xi_2+\gamma\xi_3,$$

where α, β, γ are chosen in any manner such that the determinant of the substitution is unity, viz.:

$$\alpha \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} + \beta \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} + \gamma \begin{vmatrix} A & B \\ A' & B' \end{vmatrix} = 1.$$

By hypothesis the determinants are not all zero, so that solutions α, β, γ in the $GF[p^n]$ certainly exist.

4. By application of the general theorem of §2 of the earlier paper, a ternary homogeneous substitution of determinant unity in the $GF[p^n]$ can be reduced by a linear transformation of indices to one of the following five types of canonical forms:[*]

$$x' = \lambda x, \quad y' = \lambda^{p^n} y, \quad z' = \lambda^{p^{2n}} z \qquad [\lambda^{p^{2n}+p^n+1} = 1] \quad (1)$$

arising when the characteristic determinant

$$\Delta(\lambda) \equiv \lambda^3 - \alpha\lambda^2 + \beta\lambda - 1$$

is irreducible in the $GF[p^n]$, its (imaginary) roots being, therefore, $\lambda, \lambda^{p^n}, \lambda^{p^{2n}}$.

$$x' = \mu x, \quad y' = \mu^{p^n} y, \quad z' = \mu^{-(p^n+1)} z, \qquad (2)$$

arising when $\Delta(\lambda)$ has a quadratic irreducible factor with roots μ, μ^{p^n} and a linear factor, whose root must evidently be the reciprocal of $\mu \cdot \mu^{p^n}$, and, therefore, belongs to the $GF[p^n]$.

For the remaining types, the roots of $\Delta(\lambda) = 0$ all belong to the $GF[p^n]$. In the case of multiple roots, there arises more than one type of canonical form

$$\begin{array}{llll} x' = \alpha x, & y' = \beta y, & z' = \gamma z, & [\alpha\beta\gamma = 1], \quad (3) \\ x' = \alpha x, & y' = \beta y, & z' = \beta(z + y), & [\alpha\beta^2 = 1], \quad (4) \\ x' = \alpha x, & y' = \alpha(y + x), & z' = \alpha(z + y), & [\alpha^3 = 1]. \quad (5) \end{array}$$

In the last form we may set $\alpha = 1$ when applying our results to the group G.

---

[*] An interchange of the indices does not give rise to a new type, e. g.,

$$x' = \alpha x, \quad y' = \alpha(y + x), \quad z' = \beta z, \qquad [\alpha\beta^2 = 1].$$

Indeed, we can always make a new transformation of indices of determinant unity which interchanges any two indices and at the same time changes the signs of all three indices.

It will be convenient to treat the type (4) in two cases, according as $\alpha = \beta$ or $\alpha \neq \beta$, the order of (4) differing in the two cases.

If two substitutions S and T belong to the $GF[p^n]$ and have the same canonical form, there exists (by §8 of the earlier paper) a substitution W belonging to the field such that $T = W^{-1} S W$. It remains to consider whether or not there exists in the field a substitution W_1 of determinant unity which transforms S into T. Let w be the determinant of W.

For the canonical forms (1), (2), (3) and (4), it will be shown that each canonical form can be transformed into itself (retaining the same properties concerning the conjugacy of its indices) by a substitution V of determinant equal to an arbitrary mark $\neq 0$ of the field, in particular, one of determinant w^{-1}. Expressing it in the initial indices, we obtain a substitution V_1 belonging to the field, and of determinant w^{-1}, and, finally, transforming S into itself. Hence, the product $V_1 W$ will be the required substitution of determinant $w^{-1}. w = 1$, which belongs to the field and transforms S into T. Hence, will two substitutions in the $GF[p^n]$, which have the same canonical form (1), (2), (3) or (4), be conjugate within the group of substitutions in the field and having determinant unity.

For the type (1), we may take as V the substitution

$$x' = \sigma^r x, \quad y' = \sigma^{rp^n} y, \quad z' = \sigma^{rp^{2n}} z,$$

where σ is a primitive root in the $GF[p^{3n}]$, so that

$$\rho \equiv \sigma^{1+p^n+p^{2n}}$$

is a primitive root in the $GF[p^n]$. The determinant of V is, therefore, ρ^r, which, by proper choice of r, may be made equal to an arbitrary mark $\neq 0$ of the $GF[p^n]$.

For the types (2) and (3), we may take V to be

$$x' = x, \quad y' = y, \quad z' = \rho^r z.$$

For the type (4), we may take as V the substitution

$$x' = \rho^r x, \quad y' = y, \quad z' = z.$$

For the type (5), there arise two cases. If $d = 1$, so that 3 is prime to $p^n - 1$, every mark in the $GF[p^n]$ is a cube. Hence, we can determine r so that ρ^{3r} shall take an arbitrary value except zero in the field. Hence, we can take V to be

$$x' = \rho^r x, \quad y' = \rho^r y, \quad z' = \rho^r z.$$

But, for $d = 3$, only $(p^n - 1)/3$ of the marks $\neq 0$ are cubes,[*] their products by β and β^2 being not-cubes, if β be any particular not-cube. We can evidently determine V such that T is the transformed of S by a substitution $V_1 W$ belonging to the field and having as determinant 1, β or β^2. There remain three types

$$x' = x, \quad y' = y + x, \quad z' = z + y; \tag{5_1}$$

$$x' = x, \quad y' = y + \beta x, \quad z' = z + y; \tag{5_2}$$

$$x' = x, \quad y' = y + \beta^2 x, \quad z' = z + y; \tag{5_3}$$

all conjugates by means of a substitution in the field, but of determinant $\neq 1$. In fact, the substitution

$$B: \quad \bar{x} = \beta x, \quad \bar{y} = y, \quad \bar{z} = z$$

transforms (5_2) into (5_1), while B^2 transforms (5_3) into (5_1). The most general substitution which transforms (5_1) into (5_2) is seen to be[†]

$$A: \quad x' = \frac{c}{\beta} x, \quad y' = cy + bx, \quad z' = cz + by + ax,$$

of determinant c^3/β, which cannot be unity. The most general substitution which transforms (5_2) into (5_3) is

$$A': \quad x' = \frac{c}{\beta} x, \quad y' = cy + \beta bx, \quad z' = cz + by + \beta ax,$$

gotten by transforming A by B^{-1}. But A' has determinant $c^3/\beta \neq 1$. Hence, for our group G, the types (5_1), (5_2), (5_3) are all distinct.

---

[*] Annals of Mathematics, vol. XI, 1897, p. 176.

[†] By §9 of the earlier paper, it is impossible to transform (5_1) into $\Theta(5_2)$. We can verify this result otherwise. If S be a general substitution replacing x by $ax + by + cz$, the products $(5_1)S$ and $\Theta S(5_2)$ will replace x by the same function only when

$$\theta c = c, \quad \theta b = b + c, \quad \theta a = a + b.$$

Since $\theta \neq 1$, $c = 0$, and hence $b = 0$ and, finally, $a = 0$. But this is impossible.

5. Type (1). The substitution of determinant unity

$$\begin{pmatrix} \alpha & -\beta & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has the characteristic determinant

$$\Delta(\lambda) \equiv \lambda^3 - \alpha\lambda^2 + \beta\lambda - 1.$$

Hence, there exist homogeneous linear substitutions whose characteristic determinant has the middle coefficients α and β arbitrary marks in the $GF[p^n]$, and, therefore, one whose root λ is a primitive root of the equation

$$\lambda . \lambda^{p^n} . \lambda^{p^{2n}} = 1.$$

The order of the corresponding substitution (1) is the least integer m for which

$$\lambda^m = \lambda^{mp^n} = \lambda^{mp^{2n}},$$

i. e., for which $m(p^n - 1)$ is a multiple of $p^{2n} + p^n + 1$. But the greatest common divisor of $p^n - 1$ and $p^{2n} + p^n + 1$ is also that of $p^n - 1$ and 3, and, therefore, equals d. The order m is consequently $\frac{1}{d}(p^{2n} + p^n + 1)$.

Moreover, the roots of any irreducible cubic of the form $\Delta(\lambda) = 0$ are of the form λ^s, λ^{sp^n}, $\lambda^{sp^{2n}}$, so that the corresponding substitution is the s^{th} power of the substitution just considered. Hence, the orders of all substitutions having irreducible characteristic determinants are factors of $\frac{1}{d}(p^{2n} + p^n + 1)$.

Consider a substitution S of type (1) for which λ is a primitive root of

$$\lambda^{p^{2n}+p^n+1} = 1. \tag{a}$$

By §§5 and 9 of the earlier paper, the only substitutions of G which are commutative with S have the canonical form (simultaneously with the canonical form (1) of S):

$$x' = \sigma^r x, \quad y' = \sigma^{rp^n} y, \quad z' = \sigma^{rp^{2n}} z, \qquad (\sigma^{r(1+p^n+p^{2n})} = 1),$$

where σ is a primitive root of the $GF[p^{3n}]$. Hence, $r(1 + p^n + p^{2n})$ must be divisible by $p^{3n} - 1$, and, therefore, r divisible by $p^n - 1$. Setting $r = \rho(p^n - 1)$,

$$\sigma^r = (\sigma^{p^n - 1})^\rho = \lambda^{t\rho},$$

since σ^{p^n-1} is a primitive root of (a) and hence equal to some power t of λ. The only substitutions of G which are commutative with S are, therefore, the powers of S. It follows that S is one of a set of

$$s \equiv \frac{N}{1/d\,(p^{2n}+p^n+1)}$$

distinct conjugate substitutions, N being the order of G.

The only distinct powers of S which have the same characteristic determinant as S are evidently S, S^{p^n} and $S^{p^{2n}}$. To each set of three substitutions such as S^r, S^{rp^n}, $S^{rp^{2n}}$ contained in the cyclic group generated by S and all belonging to the same characteristic determinant, there corresponds a set of s distinct conjugate substitutions. Hence, there exist in G

$$\tfrac{1}{3}\left[\frac{1}{d}\,(p^{2n}+p^n+1)-1\right]$$

such sets of conjugate substitutions. It follows that G contains in all

$$\frac{dN}{3\,(p^{2n}+p^n+1)}\left[\frac{1}{d}\,(p^{2n}+p^n+1)-1\right] \tag{6}$$

substitutions not the identity whose orders are factors of $\frac{1}{d}\,(p^{2n}+p^n+1)$. Hence, there are $\frac{dN}{3\,(p^{2n}+p^n+1)}$ distinct cyclic subgroups of order $\frac{1}{d}\,(p^{2n}+p^n+1)$, all conjugate under G. Each must, therefore, be contained self-conjugately in a subgroup of G of order $\frac{3}{d}\,(p^{2n}+p^n+1)$.

6. Type (2). Since G contains substitutions in whose characteristic determinant $\Delta(\lambda) \equiv \lambda^3 - \alpha\lambda^2 + \beta\lambda - 1$, both α and β are arbitrary in the $GF[p^n]$, we can choose

$$\alpha \equiv \gamma + 1/\delta, \quad \beta \equiv \delta + \gamma/\delta,$$

so that

$$\Delta(\lambda) \equiv (\lambda - 1/\delta)(\lambda^2 - \gamma\lambda + \delta),$$

where γ and δ are arbitrary in the $GF[p^n]$. In particular, G contains a substitution T whose characteristic determinant has an irreducible quadratic factor which vanishes for a primitive root μ of the $GF[p^{2n}]$. The canonical form of T

is then (2). The order of T is, therefore, the least integer t for which

$$\mu^t = \mu^{tp^n} = \mu^{-t(p^n+1)},$$

i. e., for which both $t(p^n - 1)$ and $t(p^n + 2)$ are divisible by $p^{2n} - 1$. But $3t$ and $t(p^n - 1)$ are both divisible by $p^{2n} - 1$, for t a minimum, if and only if,

$$t = p^{2n} - 1, \text{ when } p^n = 3^n \text{ or } 3l - 1; \; t = \tfrac{1}{3}(p^{2n} - 1), \text{ when } p^n = 3l + 1.$$

Hence, the order of T is $\frac{1}{d}(p^{2n} - 1)$.

By §§5 and 9 of the earlier paper, the most general substitution of G, which is commutative with T, has the canonical form

$$x' = \mu^r x, \quad y' = \mu^{rp^n} y, \quad z' = \sigma z. \tag{c}$$

The determinant of (c) being unity,

$$\sigma = \mu^{-r(p^n+1)}.$$

Hence, (c) reduces to T^r. It follows that T is one of a set of $\frac{dN}{p^{2n}-1}$ distinct conjugate substitutions. The only distinct powers of S which have the same multipliers as S are clearly S and S^{p^n}. Hence G contains $\frac{1}{2}\,\frac{dN}{p^{2n}-1}$ distinct conjugate cyclic subgroups of order $\frac{1}{d}(p^{2n} - 1)$, each of which is thus contained self-conjugately in a subgroup of order $\frac{2}{d}(p^{2n} - 1)$.

The number of substitutions of G whose orders are factors of $\frac{1}{d}(p^{2n} - 1)$, without being at the same time factors* of $\frac{1}{d}(p^n - 1)$, is $\frac{1}{2}\,\frac{Np^n}{p^n + 1}$. In fact, such substitutions form

$$\frac{1}{2d}\left[(p^{2n} - 1) - (p^n - 1)\right] \equiv \frac{1}{2d}(p^n - 1)p^n$$

different sets, those in each set having the same characteristic determinant. It was shown above that each such set contains $\frac{dN}{p^{2n}-1}$ distinct conjugate substitutions. The product gives the total number of such substitutions in G:

---

* If $d = 3$, the order is not a factor of $p^n - 1$, since the latter does not divide $\frac{1}{3}(p^{2n} - 1)$ when $p^n = 3l + 1$.

$$\frac{1}{2d}(p^n-1)p^n \cdot \frac{dN}{p^{2n}-1} \equiv \tfrac{1}{2}\frac{Np^n}{p^n+1}. \tag{7}$$

7. Type (4), for $\alpha \neq \beta$. Changing the notation, we consider the substitution P,

$$x' = \alpha(x+y), \quad y' = \alpha y, \quad z' = \alpha^{-2}z, \qquad (\alpha^3 \neq 1)$$

where α is a primitive root in the $GF[p^n]$. It generates a cyclic group of order $\frac{1}{d}p(p^n-1)$. If $p^n = 2$ or 2^2, we have $\alpha^3 = 1$, so that these two cases are here excluded; the reasoning below would, in fact, fail, since the order P is p for these two cases.

Considered as an operation of the isomorphic permutation-group, P belongs to a subgroup of G which leaves fixed the symbols $\{y\}$ and $\{z\}$. The general substitution possessing this property has the form

$$R\colon\quad x' = \alpha x + \alpha' y + \alpha'' z, \quad y' = \beta y, \quad z' = \gamma z. \qquad [\alpha\beta\gamma = 1].$$

In order that R shall have the order $\frac{1}{d}p(p^n-1)$, it is necessary and sufficient that α be a primitive root in the $GF[p^n]$, and that either

$$\text{(i).}\ \alpha' \neq 0, \quad \alpha = \beta \neq \gamma; \text{ or (ii).}\ \alpha'' \neq 0, \quad \alpha = \gamma \neq \beta.$$

Indeed, if both β and γ differ from α, we may, by introducing in place of x, the new index

$$X \equiv x + \frac{\alpha'}{\alpha-\beta}y + \frac{\alpha''}{\alpha-\gamma}z,$$

give R the form

$$X' \equiv \alpha X, \quad y' = \beta y, \quad z' = \gamma z,$$

whose $(p^n-1)^{\text{st}}$ power is unity. If, therefore, $\alpha \neq \beta$, we may take $\alpha = \gamma$. Then $\alpha'' \neq 0$; for, if $\alpha'' = 0$, R multiplies $x + \frac{\alpha'}{\alpha-\beta}y$ by α, so that R would have as order a factor of (p^n-1). Similarly, if $\alpha \neq \gamma$, then must $\alpha = \beta$, $\alpha' \neq 0$.

Finally, if $\alpha=\beta=\gamma$, each may be taken equal to unity. Then, by induction,

$$R^r:\quad x'=x+r\alpha' y+r\alpha'' z,\quad y'=y,\quad z'=z,$$

so that R is of period p. Hence, either (i) or (ii) are necessary conditions.

Consider the case when the relations (i) are satisfied. Setting

$$X\equiv x+\frac{\alpha'' z}{\beta-\gamma},\quad Y\equiv\frac{\alpha'}{\alpha}y,$$

R takes the form

$$X'=\alpha(X+Y),\quad Y'=\alpha Y,\quad z'=\alpha^{-2}z.$$

This substitution is of period $\frac{1}{d}p(p^n-1)$ if, and only if, α be a primitive root in the $GF[p^n]$.

Interchanging y with z, the proof follows for case (ii).

With the aid of the theorem just proven, we proceed to determine the number and conjugacy of the cyclic groups of order $\frac{1}{d}p(p^n-1)$, which leave the symbols $\{y\}$ and $\{z\}$ fixed. Consider first the case (i),

$$\alpha'\neq 0,\quad \alpha=\beta,\quad \gamma=\alpha^{-2}\neq\alpha,\qquad \alpha=\text{primitive root in the } GF[p^n]\,;$$
$$R:\quad x'=\alpha x+\alpha' y+\alpha'' z,\quad y'=\alpha y,\quad z'=\alpha^{-2}z.$$

By simple induction we verify that R^t has the form

$$x'=\alpha^t x+t\alpha'\alpha^{t-1}y+\alpha''\alpha^{t-1}\left(\frac{\alpha^{-3t}-1}{\alpha^{-3}-1}\right)z,\quad y'=\alpha^t y,\quad z'=\alpha^{-2t}z.$$

In order that ΘR^t shall be identical with

$$x'=\alpha x+\rho' y+\rho'' z,\quad y'=\alpha y,\quad z'=\alpha^{-2}z,$$

it is necessary and sufficient that

$$\theta\alpha^{t-1}=1,\quad t\alpha'=\rho',\quad \alpha''=\rho''.$$

It follows that the set of $p^n.\frac{p^n-1}{p-1}$ distinct substitutions

$$x'=\alpha x+My+\alpha'' z,\quad y'=\alpha y,\quad z'=\alpha^{-2}z, \tag{d}$$

where α is a fixed mark $\neq 0$, α'' an arbitrary mark, and M any one of the $\frac{p^n-1}{p-1}$ distinct marks M_1, M_2, ... such that no two have as their ratio an *integral* mark,* has the property that no power of one of the substitutions (d) equals another substitution of the set. We, therefore, obtain $p^n(p^n-1)/(p-1)$ distinct cyclic subgroups of order $\frac{1}{d}p(p^n-1)$.

Furthermore, every substitution V of the subgroup leaving $\{y\}$ and $\{z\}$ fixed, and having $\alpha=\beta$, and lastly of order a factor of $\frac{1}{d}p(p^n-1)$ without being a factor of p or (p^n-1), is contained in one of these cyclic subgroups. In proof, we observe that, by the argument used at the beginning of the paragraph, we may set

$$V:\quad x'=\alpha^s x+\alpha' y+\alpha'' z,\quad y'=\alpha^s y,\quad z'=\alpha^{-2s}z,\quad (\alpha'\neq 0,\ \alpha^{3s}\neq 1).$$

Let M_i be such a mark that its ratio to α'/α^{s-1} is an integral mark. Then the power $s+k(p^n-1)$ of the substitution of the form (d) above

$$x'=\alpha x+M_i y+Az,\quad y'=\alpha y,\quad z'=\alpha^{-2}z,$$

is

$$x'=\alpha^s x+[s+k(p^n-1)]\,\alpha^{s-1}M_i y+A\alpha^{s-1}\left(\frac{\alpha^{-3s}-1}{\alpha^{-3}-1}\right)z,\quad y'=\alpha^s y,\quad z'=\alpha^{-2s}z.$$

By choice of k, we can make

$$[s+k(p^n-1)]\,\alpha^{s-1}M_i=\alpha',$$

and, by choice of A, we can make the coefficient of z in x' equal to α''. It follows that there are $p^n(p^n-1)/(p-1)$ cyclic subgroups of order $\frac{1}{d}p(p^n-1)$ for which $\alpha=\beta$, and as many more for which $\alpha=\gamma$, each leaving the symbols $\{y\}$ and $\{z\}$ fixed, and together containing all substitutions of the last property having an order not p nor a factor of p^n-1.

These cyclic subgroups are all conjugate within G, and, indeed, within the subgroup which leaves $\{y\}$ and $\{z\}$ fixed or permutes them. In proof, we first observe that the cases for which $\alpha^{-2}=\alpha$, viz., $p^n=2$ and $p^n=4$, have been excluded. We verify that the substitution

$$x'=x+\frac{B-A}{\alpha^{-2}-\alpha}z,\quad y'=y,\quad z'=z$$

---

* The marks M_1, M_2, ... are evidently the multipliers in a rectangular array of the marks $\neq 0$ of the $GF[p^n]$, the first row being formed by the integral marks 1, 2, ..., $p-1$.

transforms the substitution

$$x' = \alpha x + My + Az \quad , \quad y' = \alpha y, \quad z' = \alpha^{-2} z$$

into $\quad x' = \alpha x + My + Bz \quad , \quad y' = \alpha y, \quad z' = \alpha^{-2} z.$

Further, $\quad x' = \lambda\rho x \quad , \quad y' = \rho y, \quad z' = \lambda^{-1}\rho^{-2} z$

transforms $\quad x' = \alpha x + My + Az \quad , \quad y' = \alpha y, \quad z' = \alpha^{-2} z$

into $\quad x' = \alpha x + \lambda My + \lambda^2\rho^3 Az, \quad y' = \alpha y, \quad z' = \alpha^{-2} z.$

Hence the cyclic subgroups given by $\alpha = \beta$ are all conjugate within the group, leaving $\{y\}$ and $\{z\}$ fixed.

The substitution

$$x' = x \quad , \quad y' = -z \quad , \quad z' = y$$

interchanges $\{y\}$ with $\{z\}$ and transforms

$$x' = \alpha x + My + Az, \quad y' = \alpha y \quad , \quad z' = \alpha^{-2} z$$

into $\quad x' = \alpha x - Ay + Mz, \quad y' = \alpha^{-2} y, \quad z' = \alpha z.$

Hence the set of cyclic groups given by $\alpha = \beta$ are conjugate to the set given by $\alpha = \gamma$ within the group, leaving fixed $\{y\}$ and $\{z\}$ or permuting them. The latter group, therefore, contains $2p^n(p^n - 1)/(p - 1)$ conjugate cyclic groups of order $\frac{1}{d}\, p\, (p^n - 1)$, and those substitutions of these groups whose orders are not divisors of p or $(p^n - 1)$ are all distinct. But the general permutation-group is doubly transitive, and hence contains

$$\tfrac{1}{2}\,(p^{2n} + p^n + 1)(p^{2n} + p^n)$$

conjugate subgroups, leaving fixed two symbols or permuting them. In all, we have

$$2p^n\left(\frac{p^n - 1}{p - 1}\right) \cdot \tfrac{1}{2}\,(p^{2n} + p^n + 1)(p^{2n} + p^n) \equiv \frac{dN}{p^n\,(p^n - 1)(p - 1)}.$$

conjugate cyclic subgroups of order $\frac{1}{d}\, p\, (p^n - 1)$. Each is, therefore, contained self-conjugately in a subgroup of order $\frac{1}{d}\, p^n\, (p^n - 1)(p - 1)$.

Each cyclic subgroup contains $p+\frac{1}{d}(p^n-1)-1$ substitutions of order p or a divisor of $\frac{1}{d}(p^n-1)$, the latter giving all of its substitutions whose orders divide p^n-1. There remain $(p-1)\left[\frac{1}{d}(p^n-1)-1\right]$ substitutions. Hence, G contains

$$\frac{N(p^n-1-d)}{p^n(p^n-1)} \tag{8}$$

substitutions whose orders divide $\frac{1}{d}p(p^n-1)$ but not p or p^n-1. For the cases $p^n=2$ or 2^2 above excluded, formula (8) reduces to zero. The result is, therefore, true generally.

8. Type 4, when $\alpha=\beta$. We have to consider substitutions of order p of the canonical form

$$x'=x+z,\quad y'=y,\quad z'=z.$$

From the investigation at the beginning of §7, it follows that the only substitutions of period p which leave fixed the symbols $\{y\}$ and $\{z\}$ have the form

$$x'=x+\alpha y+\beta z,\quad y'=y,\quad z'=z, \tag{e}$$
$$(\alpha \text{ and } \beta \text{ not both zero.})$$

There are $p^{2n}-1$ distinct substitutions of this form. They are all conjugate within G, being reducible to the above canonical form. In fact, if $\beta\neq 0$, we transform (e) by

$$x'=x,\quad y'=y,\quad z'=z+\rho y$$

and get the substitution

$$x'=x+(\alpha-\beta\rho)y+\beta z,\quad y'=y,\quad z'=z.$$

By choice of ρ, we can make $\alpha-\beta\rho=0$. If $\beta=0$, we transform (e) by

$$x'=-x\quad,\quad y'=z,\quad z'=y,$$

giving

$$x'=x-\alpha z,\quad y'=y,\quad z'=z.$$

In either case, we reach a substitution of the form (e), but having the coefficient $\alpha = 0$, and, therefore, $\beta \neq 0$. Transforming it by

$$x' = x\ , \quad y' = \beta^{-1} y, \quad z' = \beta z,$$

we get

$$x' = x + z, \quad y' = y\ , \quad z' = z.$$

The $p^{2n} - 1$ substitutions (e) determine $(p^{2n} - 1)/(p - 1)$ conjugate cyclic subgroups of order p and contained in the subgroup, leaving fixed the symbols $\{y\}$ and $\{z\}$, and hence also $\{y + \rho z\}$, ρ being an arbitrary mark in the $GF[p^n]$. Each such group, therefore, leaves fixed $p^n + 1$ (and no more) symbols. But the $p^{2n} + p^n + 1$ symbols furnish

$$\frac{\frac{1}{2}(p^{2n} + p^n + 1)(p^{2n} + p^n)}{\frac{1}{2}(p^n + 1)p^n} \equiv p^{2n} + p^n + 1$$

such sets of symbols. Hence, G contains

$$(p^{2n} + p^n + 1)\frac{(p^{2n} - 1)}{(p - 1)} \equiv \frac{dN}{p^{3n}(p^n - 1)(p - 1)}$$

such conjugate cyclic subgroups, all of whose substitutions are conjugate under G. Each such subgroup is, therefore, contained self-conjugately within a subgroup of order $\frac{1}{d} p^{3n}(p^n - 1)(p - 1)$. The total number of distinct substitutions of G of order p of the type considered has thus been shown to be

$$\frac{dN}{p^{3n}(p^n - 1)}. \tag{9}$$

9. Type (5_1). If $p > 2$, the substitution

$$W_1: \quad x' = x + y \quad y' = y + z, \quad z' = z,$$

is of order p. The most general substitution transforming W_1 into itself has the form

$$x' = ax + by + cz, \quad y' = ay + bz, \quad z' = az. \tag{f}$$

If it have determinant unity, $a = 1$, θ or θ^2. Hence, there are dp^{2n} such substitutions. The following substitution of determinant unity

$$x' = tx, \quad y' = y - \frac{t-1}{2t} z, \quad z' = \frac{1}{t} z$$

will transform W_1 into W_1^t, viz.:

$$W_1^t\colon \quad x' = x + ty + \tfrac{1}{2} t(t-1) z, \quad y' = y + tz, \quad z' = z.$$

Taking $t = 1, 2, \ldots, p-1$, we get $dp^{2n}(p-1)$ distinct homogeneous substitutions of determinant unity which transform into itself the cyclic group generated by W_1. There correspond $p^{2n}(p-1)$ distinct substitutions in G. The cyclic group $\{W_1\}$ is, therefore, one of $\frac{N}{p^{2n}(p-1)}$ distinct conjugate subgroups of G. Hence, G contains N/p^{2n} distinct conjugate substitutions of the type (5_1).

Since (5_2) and (5_3) are conjugate to (5_1) within the general linear homogeneous group, the number of substitutions of G conjugate within G to (5_1) equals the number conjugate to (5_2) or to (5_3). Hence, there are in G together

$$3N/p^{2n} \tag*{$(10)_{p>2}$}$$

distinct substitutions of the types (5_1), (5_2) and (5_3), forming three distinct sets of conjugate subgroups.

10. Types (5_1), (5_2) and (5_3) for $p = 2$. The order of the canonical types,

$$W_i\colon \quad x' = x + y, \quad y' = y + \beta^i z, \quad z' = z \qquad (i = 0, 1, 2)$$

is now 4. Indeed, we have

$$\begin{aligned} W_i^2\colon \quad & x' = x + \beta^i z, & y' &= y, & z' &= z; \\ W_i^3\colon \quad & x' = x + y + \beta^i z, & y' &= y + \beta^i z, & z' &= z. \end{aligned}$$

Since W_i leaves fixed but one symbol $\{z\}$, while W_i^2 leaves fixed the $2^n + 1$ symbols $\{z\}$, $\{y + \lambda z\}$ ($\lambda =$ arbitrary mark of the $GF[2^n]$), the two substitutions are not conjugate under G. But W_i is transformed into W_i^3 by the substitution

$$x' = x + y, \quad y' = y, \quad z' = z.$$

As in §9, the most general substitution transforming W_i into itself is

$$x' = ax + by + cz, \quad y' = ay + b\beta^i z, \quad z' = az.$$

Its determinant must be unity, whence $a^3 = 1$. It follows that G contains just 2^{2n} distinct substitutions which transform W_i into itself and, therefore, as many more which transform W_i into W_i^3. The cyclic group of order 4 generated by W_i is, consequently, one of $N/2^{2n+1}$ conjugate subgroups of G. Just two of the substitutions of every such subgroup are of type W_i, the remaining one,* $\neq I$, being of type (4) with $\alpha = \beta$. Hence, G contains

$$\frac{3N}{2^{2n}} \qquad (10)_{p=2}$$

distinct substitutions of the types (5_1), (5_2), (5_3) for $p = 2$, all distinct from those enumerated in §8. As in the case $p > 2$, they fall into three distinct sets of conjugate substitutions under G.

11. Type (3). The substitutions of the canonical form

$$x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z \qquad [\alpha\beta\gamma = 1] \qquad (3)$$

are of order a divisor of $p^n - 1$. Of the $(p^n - 1)^2$ sets of solutions in the $GF[p^n]$ of $\alpha\beta\gamma = 1$, d sets have $\alpha = \beta = \gamma$, and hence each equal to θ^r ($r = 0$, 1 or 2). If α be any mark different from 0, 1, θ, θ^2, and if $\beta = \alpha$, then $\gamma = \alpha^{-2} \neq \alpha$. Hence, there are $3(p^n - d - 1)$ sets of solutions, in which two, and only two, of the quantities α, β, γ are equal. There remain

$$(p^n - 1)^2 - 3(p^n - d - 1) - d \equiv p^{2n} - 5p^n + 4 + 2d$$

sets of solutions in which α, β, γ are all distinct. Dividing this number by 6 to allow for permutations, we obtain the number of distinct sets of unequal multipliers of ternary homogeneous substitutions (3).

If, for $d = 3$, α, β, γ do not form a permutation of 1, θ, θ^2, the three sets

$$\alpha, \beta, \gamma; \quad \theta\alpha, \theta\beta, \theta\gamma; \quad \theta^2\alpha, \theta^2\beta, \theta^2\gamma,$$

---

[*] W_1^2 and W_2^2 are readily transformed into W_0^2. For example, the substitution

$$x' = x, \quad y' = \beta^{-1}y, \quad z' = \beta z$$

of determinant unity, transforms W_1^2 into W_0^2.

are not equivalent sets of multipliers in the homogeneous group, but are equivalent in the non-homogeneous group G. The number of sets of unequal multipliers in G is, therefore,

$$1+\tfrac{1}{3}\left(\frac{p^{2n}-5p^n+4+2d}{6}-1\right),\ \text{for } d=3;\quad \frac{p^{2n}-5p^n+4+2d}{6},\ \text{for } d=1.$$

By §5 of the earlier paper, the only homogeneous substitutions commutative with (3), for α, β, γ distinct, are the $(p^n-1)^2$ substitutions

$$T:\quad x'=ax,\quad y'=by,\quad z'=cz,\qquad (abc=1).$$

By §9 of the earlier paper, there exist substitutions transforming S, given by (3), into ΘS only when α, β, γ form a permutation of 1, θ, θ^2. In the latter case, S has one of the forms T, PT, P^2T, where P denotes the substitution

$$P:\quad x'=y,\quad y'=z,\quad z'=x.$$

In this case, there are $3(p^n-1)^2$ homogeneous substitutions commutative with (3), and, therefore, $(p^n-1)^2$ substitutions of G commutative with the substitutions corresponding to (3) in G. Hence,

$$x'=x,\quad y'=\theta y,\quad z'=\theta^2 z$$

is one of a set $\dfrac{N}{(p^n-1)^2}$ distinct conjugate substitutions under G.

For α, β, γ distinct and not a permutation of 1, θ, θ^2, each substitution (3) is one of a set of $\dfrac{N}{1/d\,(p^n-1)^2}$ conjugate substitutions under G. The total number of such substitutions is, therefore,

$$\frac{dN}{(p^n-1)^2}\cdot\tfrac{1}{3}\left(\frac{p^{2n}-5p^n+4+2d}{6}-1\right),\ \text{for } d=3;$$

$$\frac{N}{(p^n-1)^2}\left(\frac{p^{2n}-5p^n+4+2d}{6}\right),\ \text{for } d=1.$$

Combining our results, the total number of substitutions of G of canonical form (3) in which α, β, γ are distinct, is for $d=1$ or 3:

$$\frac{N}{(p^n-1)^2}\cdot\frac{p^{2n}-5p^n+4+2d}{6}. \tag{11}$$

Consider next the $p^n - d - 1$ sets of multipliers α, β, γ, two of which are equal. There correspond to the substitutions (3) $1/d\,(p^n - d - 1)$ substitutions of G, no two of which are conjugate, having different sets of multipliers. The substitution

$$A: \quad x' = \alpha x, \quad y' = \alpha y, \quad z' = \gamma z \qquad [\alpha^2\gamma = 1, \ \gamma \neq \alpha]$$

cannot, by §9 of the earlier paper, be transformed into ΘA. The most general substitution transforming A into itself is, by §5 of the earlier paper,

$$x' = ax + by, \quad y' = a'x + b'y, z' = c''z.$$

The number of such substitutions of determinant unity is

$$(p^{2n} - 1)(p^{2n} - p^n).$$

Hence, the total number of substitutions of G having the canonical form A is

$$\frac{1}{d}(p^n - d - 1) . \frac{N}{1/d(p^{2n}-1)(p^{2n}-p^n)} \equiv \frac{N(p^n - d - 1)}{(p^{2n} - 1)(p^{2n} - p^n)} . \qquad (12)$$

12. As a check upon the accuracy of our enumeration of the substitutions of G, we may verify that the numbers given by the formulæ (6), (7), (8), (9), (10), (11) and (12), together with unity (to count the identical substitution), give as total sum the number N.

13. In determining the cyclic groups generated by the substitutions of type (3), we consider in turn the cases $d = 1$ and $d = 3$. If α be a primitive root of the $GF[p^n]$, any substitution C of the form (3) may be written

$$C: \quad x' = \alpha^r x, \quad y' = \alpha^s y, \quad z' = \alpha^{-r-s} z,$$

where r and s are integers chosen from the series 0, 1, 2, ..., $p^n - 2$. Then, for $d = 1$, C is of period $p^n - 1$ if, and only if, the greatest common divisor of r, s, and $p^n - 1$ is unity, or, symbolically, $[r, s, p^n - 1] = 1$. The number of sets of integers r, s satisfying this condition is (Jordan, "Traité," p. 96)

$$F(p^n - 1) \equiv \phi(p^n - 1)\,\psi(p^n - 1) \equiv (p^n - 1)^2\left(1 - \frac{1}{q_1^2}\right)\left(1 - \frac{1}{q_2^2}\right) \cdots\cdot ,$$

where q_1, q_2, ... are the distinct prime factors of $p^n - 1$. For $\phi(p^n - 1)$ values of r, the pairs r, r; $r, -2r$; $-2r, r$ are included in these sets, but lead to only $\phi(p^n - 1)$ sets of multipliers in C. The remaining $F - 3\phi(p^n - 1)$ sets r, s lead to $\frac{1}{6}\{F(p^n - 1) - 3\phi(p^n - 1)\}$ sets of unequal multipliers in C.

The substitutions (3) all lie in the cyclic groups generated by substitutions C of period p^n-1. In certain of these cyclic groups, the $\phi(p^n-1)$ substitutions of period p^n-1 have in pairs the same set of multipliers; in others they have by threes the same set of multipliers.* If C have the multipliers α, α^m, α^{-1-m}, where $m^2\equiv 1 \pmod{p^n-1}$, then C^m has the same multipliers in a different order. But if $m^2+m+1\equiv 0 \pmod{p^n-1}$, then C, C^m, C^{-1-m} all have the same set of multipliers. The first congruence has $2^{\mu+\kappa}$ solutions m, where μ denotes the number of distinct odd prime factors of p^n-1; while, if 2^k is the highest power of 2 contained in p^n-1, $\kappa=0$ if $k=0$ or 1, $\kappa=1$ if $k=2$, $\kappa=2$ if $k\geqq 3$ (Dirichlet, "Zahlentheorie," §37). The second congruence $m^2+m+1\equiv 0$ is seen† to have solutions for $d=1$ only when $p^n=2^n$, n odd, such that the prime factors (say γ distinct ones) are all of the form $6j+1$. But if m be a solution, so is also $-1-m$, giving but $2^{\gamma-1}$ sets of multipliers α, α^m, α^{-1-m}. The solutions $m>1$ of $m^2\equiv 1$ lead to $2^{\mu+\kappa}-1$ distinct cyclic groups of order p^n-1, such that the sets of multipliers of their substitutions of period p^n-1 are the same in pairs, and containing in all $\frac{1}{2}\phi(p^n-1)(2^{\mu+\kappa}-1)$ distinct sets of multipliers of substitutions of period p^n-1. The solutions of $m^2+m+1\equiv 0$ lead to $2^{\gamma-1}$ distinct cyclic groups of order p^n-1, containing $\frac{1}{3}\phi(p^n-1)\,2^{\gamma-1}$ distinct sets of multipliers of substitutions of period p^n-1, those in each cyclic group having coincided in sets of three. Denote by $\rho\phi(p^n-1)$ the combined number of sets of multipliers in these two classes of cyclic groups. In every other cyclic subgroup, the sets of multipliers of the substitutions of period p^n-1 are found to be distinct. Hence, the substitutions (3) generate the following classes of non-conjugate cyclic groups of order p^n-1:

(i). $2^{\mu+\kappa}-1$ groups generated by substitutions with multipliers α, α^m, α^{-1-m} with $m^2\equiv 1 \pmod{p^n-1}$, $m>1$.

(ii). $2^{\gamma-1}$ groups with similar generators having $m^2+m+1\equiv 0 \pmod{p^n-1}$.

(iii). One group generated by the substitution with multipliers α, α, α^{-2}.

(iv). $\frac{1}{6}\{\psi(p^n-1)-3\}-\rho$ groups generated by substitutions with unequal multipliers and not conjugate with any of their powers.

A cyclic group of class (i), (ii), (iii) or (iv) is transformed into itself by the following number of substitutions of G respectively:

$$2(p^n-1)^2,\quad 3(p^n-1)^2,\quad (p^{2n}-1)(p^{2n}-p^n)\ \text{or}\ (p^n-1)^2.$$

---

* The statements of Burnside, l. c., middle of p. 77, are not exact.

† Using Dirichlet, §§85 and 87, and Gauss, Disq. Arith., Art. 120.

In fact, each is commutative with the $(p^n-1)^2$ substitutions C; class (iv) with no other substitutions of G; class (i) also with CT, where T replaces x by y and y by $-x$; class (ii) also with $(xyz)\,C$ and $(xzy)\,C$; class (iii) with

$$x' = ax + by, \quad y' = cx + dy; \quad z' = ez.$$

14. For $d = 3$, set $p^n - 1 = 3t$. We may establish the theorem:*

If t be prime to 3, every substitution (3) *is some power of a substitution* (3) *of period $p^n - 1$; if t be divisible by 3, no cyclic group of order $p^n - 1$ generated by a substitution* (3) *contains one of the substitutions of period t*

$$x' = \alpha x, \quad y' = \alpha^{3l+1}y, \quad z' = \alpha^{-3l-2}z.$$

Except when $p^n - 1$ is divisible by 9, it, therefore, suffices to study the cyclic groups of order $p^n - 1$. The substitution C is of period $p^n - 1$, if, and only if, $[r, s, p^n - 1] = 1$ and $r - s$ be prime to 3. Denote by M the number of sets of multipliers giving distinct substitutions C of period $p^n - 1$ in the quotient-group G. We can readily prove that, if t be prime to 3, $M = \frac{1}{3}F(t)$; while, if t be divisible by 3 and we set $t \equiv T3^\tau$, T being prime to 3, then $M = 3^{2\tau-1}F(T)$.

Supposing first that t is prime to 3, we may establish the following complete list of non-conjugate cyclic groups of order $p^n - 1$ generated by the substitutions (3):

(a). $2^{\mu+\kappa-1}$ groups generated by substitutions with multipliers α, α^m, α^{-1-m}, where $m^2 \equiv 1 \pmod{3t}$, $m \equiv -1 \pmod 3$.

(b). $2^{\delta-1}$ groups generated by similar substitutions with $m^2 + m + 1 \equiv 0 \pmod{3t}$, $m \equiv 1 \pmod 3$, occurring only when $p^n = 2^n$, n even and prime to 3, such that $\frac{1}{3}(2^n - 1)$ has only prime factors (δ distinct ones) of the form $6j + 1$.

(c). $\dfrac{1}{\phi(3t)}\{\frac{1}{3}F(t) - \frac{1}{2}\phi(3t)\,2^{\mu-1+\kappa} - \frac{1}{3}\phi(3t)\,2^{\delta-1}\}$ groups generated by substitutions of period $p^n - 1$ not conjugate with any of their powers.

If t be divisible by 3, the only cyclic groups of order $p^n - 1$ are:

(a). $2^{\mu+\kappa-1}$ groups generated by substitutions with the multipliers α, α^m, α^{-1-m}, where $m^2 \equiv 1 \pmod{3t}$, $m \equiv -1 \pmod 3$.

(b). $\dfrac{1}{\phi(3t)}\{3^{2\tau-1}F(T) - 2^{\mu+\kappa-1}.\frac{1}{2}\phi(3t)\}$ groups generated by substitutions of period $p^n - 1$ not conjugate with any of their powers.

---

* The statements of Burnside, l. c., p. 102, are not complete.

15. For the case $p^n = 2^3$, we have a simple group G of order $N \equiv 20160$. Applying the above general results to this case, G contains

	960	conjugate	cyclic groups	of order	7	with	5760	substitutions	period	7
	2016	"	"	"	5	"	8064	"	"	5
Three sets of	630	"	"	"	4	"	3.1260	"	"	4
	315	"	"	"	2	"	315	"	"	2
	1120	"	"	"	3	"	2240	"	"	3
							1	identity.		
							20160			

The substitutions of period 2 are all contained in the cyclic groups of order 4.

For comparison, we give a table of the types of substitutions in the alternating group on 8 letters:

Type.	Period.	Number in G.
(1234567)	7	5760
(123456)(78)	6	3360
(12345)	5	1344
(12345)(678)	15	2688
(1234)(56)	4	2520
(1234)(5678)	4	1260
(123)	3	112
(123)(456)	3	1120
(123)(45)(67)	6	1680
(12)(34)	2	210
(12)(34)(56)(78)	2	105
identity	1	1
		20160

The two groups differ in structure in many respects. They contain the same number of substitutions of period 7, the same number of period 4 and the same number of period 2.

NOTE.—Page 232, line 6: For § 13 read § 15.

UNIVERSITY OF TEXAS, *December* 12, 1899.

On some Invariant Scrolls in Collineations which leave a Group of Five Points Invariant.

By Virgil Snyder.

The quadric surfaces which are left invariant by cyclical collineations have been quite exhaustively studied.*

Another quite simple series of scrolls, namely, certain ones contained in a linear congruence, have equally interesting properties, but have not been heretofore considered, except one form which was noticed by Ameseder. The following note will be restricted to such surfaces. There are six collineations which are of essentially different type that project a set of five points into themselves without leaving every point invariant.

Using the notation of substitution-groups, these six collineations may be represented as follows:

$$\begin{aligned} T_2 &\equiv (A_1\,A_2)(A_3)(A_4)(A_5), \\ T_3 &\equiv (A_1\,A_2)(A_3\,A_4)(A_5), \\ T_4 &\equiv (A_1\,A_2\,A_3)(A_4)(A_5), \\ T_5 &\equiv (A_1\,A_2\,A_3)(A_4\,A_5), \\ T_6 &\equiv (A_1\,A_2\,A_3\,A_4)(A_5), \\ T_7 &\equiv (A_1\,A_2\,A_3\,A_4\,A_5). \end{aligned}$$

It is not necessary to count the number of ways in which any T_k may be represented; T_k, T_m are independent of each other in the sense that each leaves a characteristic configuration invariant, but all the invariant forms which arise from permutation of the elements in a particular T_k are projectively equivalent.

---

* J. Lüroth, "Das Imaginäre in der Geometrie und das Rechnen mit Würfen," Math. Annalen, vol. XI, p. 84, and "Ueber cyclisch-projectivische Punktgruppen in der Ebene und im Raume," ibid., vol. XIII, p. 304. A. Ameseder, "Theorie der cyclischen Projectivitäten," Wiener Sitzungsberichte (Math. natw. Classe), vol. XCVIII, IIa, p. 290, and "Die Quintupellage collinearer Räume," ibid., p. 588. R. Sturm, "Ueber Collineationen und Correlationen welche Flächen 2 Grades oder cubische Raumcurven in sich selbst transformieren," Math. Ann., vol. XXV, p. 465. H. Schroeter, "Cyclisch projective Punktquadrupel," Math. Ann., vol. XX, p. 231. H. Küppers, "Collineation durch welche fünf Punkte des Raumes . . . ." Diss. Münster, 1890.

These collineations will be considered in turn.

Let $T_2 \equiv (A_1 A_2)(A_3)(A_4)(A_5)$ be the symbol of a collineation which leaves the points A_3, A_4, A_5 invariant, and interchanges the points A_1, A_2 involutorially. The plane $A_3 A_4 A_5 \equiv \omega$ remains invariant, and likewise the line $(A_1 A_2) \equiv u$. The intersection of ω and u remains fixed; hence, every point of ω is invariant. The other invariant point on u is the harmonic conjugate of (u, ω) with regard to $A_1 A_2$.

The scrolls having u as an m-fold directrix, and any line v in ω which does not cut u as a double directrix, the two lines which issue from any point of v passing through a pair of conjugates of u, will go into themselves; the generators passing through a pair of conjugate points simply interchange; likewise, tangents belonging to positive and negative complexes simply interchange, and, consequently, the asymptotic lines belonging to positive and negative complexes also; hence, *the harmonic homology T_2 leaves a series of scrolls of the symbol* $(2, m)$ *invariant in such a way that positive and negative asymptotic lines are interchanged.*

$T_3 \equiv (A_1 A_2)(A_3 A_4)(A_5)$. This form leaves every point of each of two skew lines u, v invariant, and is known as the axial involution (Salmon-Fidler, vol. I, p. 132). The point corresponding to any point A lies on the line joining A to u and v and is the harmonic conjugate of A with regard to these two points of intersection, hence, *every scroll having u, v for directrices is left invariant by T_3 in such a way that every asymptotic line goes into itself, the two points in which it intersects each generator being interchanged.*

$T_4 \equiv (A_1 A_2 A_3)(A_4)(A_5)$. Here space is in triad position. The plane ω, containing $A_1 A_2 A_3$, is invariant and in it the point in which the line $A_4 A_5 \equiv u$ pierces it, hence, u has only invariant points. There is an invariant line in ω skew to u; call it v. The points on v are in triads; hence, the invariant points must be imaginary (Lüroth, l. c. first paper); call them R and S.

Let X_1, Y_1 be any two points in space; they generate two triads $X_1 X_2 X_3$, $Y_1 Y_2 Y_3$. Through these six points a twisted cubic curve c_3 can be drawn, which goes into itself. The points on c_3 lie in triads; hence, two imaginary invariant points lie upon it. All the self-corresponding points lie on u and v. c_3 cannot cut v, for the plane of each triad passes through v, and in that case each plane through v would cut c_3 in more than three points.

Hence, c_3 cuts u in two imaginary points OP. The points $OPRS$ are the vertices of a principal tetrahedron of T_4. The Reye complex of lines joining corresponding points breaks up into two special linear complexes with u, v for axes. The line joining any two non-coincident corresponding points must cut v.

The osculating planes of c_3 at O and P must also remain invariant; hence, they are the planes $(Ov)(Pv)$. The tangents at these points are also invariant; choose O, P such that the tangents are OR, PS. Now, consider the scroll Σ generated by all the lines which cut u, v and c_3. This will be a quartic scroll having u for a triple directrix and v for a simple one. The three lines which issue from the same point of u cut v in the points of a triad. The pinch-points are O, P, each counted twice. Let Z be any point on the surface Σ; it will generate a triad Z_1, Z_2, Z_3. Determine on a second generator g not belonging to the first triad, a point W such that the four points gu, gv, gc_3, W have the same anharmonic ratio as the points $g'u$, $g'v$, $g'c_3$, Z; Z lying on g'. Then W will generate a triad, and the locus of W, as g takes all positions on Σ, will be another twisted cubic lying on the same surface. Dually, let α, β be any two planes; they will generate two triads $\alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\beta_3$. The first triad will pass through a fixed point U_1 on u, and the second through another, U_2. These planes cut v in two triads; they uniquely determine a cubic torse $\varkappa_3$, all of whose planes are arranged in triads, and will, therefore, have two imaginary invariant planes ϕ_1, ϕ_2 through v. The points of contact are also invariant and lie on u, hence, they must be the points O, P, and the invariant planes are (Ov), (Pv). These conditions fixed c_3, hence, c_3 is self-dual, and all the generators of Σ lie in the osculating planes of c_3; these planes are also tangent planes to Σ, hence, c_3 is an asymptotic line on the surface. The other lines, loci of the points W, are the remaining asymptotic lines; hence, T_4 *transforms* ∞^6 *quartic scrolls of the type* $[3, 1]$ *into themselves in such a way that the cubic asymptotic lines go into themselves.*

The Weddle surface ψ, locus of the vertex of a quadric cone passing through two triads X, Y, goes into itself in such a way that the cubic lying in any plane through v goes into itself. The curve c_3 is an asymptotic line on this surface, hence, Σ and ψ touch each other along c_3. The ∞^6 surfaces ψ all pass through the line v, and on them it is a single line.

Each plane cubic cuts v in the points of a triad, and the points in its plane on c_3 define another triad.

The 12 Kummer surfaces which are determined by the two triads $(A_1A_2A_3)$, (B_1, B_2, B_3) of nodes go into each other in sets of fours; it is always possible to choose B_1 such that the triads form a quadruple involution on c_3; they lie in triple involution in any case. All 12 surfaces of the group are in these cases tetrahedroids, projectively equivalent to Fresnel's wave surface. The surface ψ also acquires two new lines.*

$T_5 \equiv (A_1A_2A_3)(A_4A_5)$. Since $T_5^2 = T_4$ and $T_5^3 = T_2$, it follows that only those configurations which are left invariant by both T_4 and T_2 are invariant in T_5. No simple scrolls except hyperboloids remain invariant, and in these the two systems of generators are interchanged. $T_5^6 = 1$, so that space lies in sextuple position, and it might be supposed that a twisted cubic could be passed through a sextuple, but as the points of every sextuple lie in pairs on three lines through an invariant point on account of $T_5^3 = T_2$, hence, no other cubic through these points is possible.

$T_6 \equiv (A_1A_2A_3A_4)(A_5)$. This case has already been considered in detail in the article by Schroeter mentioned above. The scrolls defined by secants of a pair of rays in involution and the invariant quartic curve, break up into two quadrics, and there are no others. The secants of this quartic and the invariant lines form the second generation of the same quadrics.

$T_7 \equiv (A_1A_2A_3A_4A_5)$. Let it be supposed that the five points are so arranged that no four lie in a plane, then two skew lines u, v remain invariant, and on each are ranges in quintuple. Then u, v have two sets of imaginary invariant points, O, P on u; R, S on v. A twisted cubic c_3 is uniquely determined by passing through the five points of a quintuple and having v for a chord; c_3 will go into itself by T_7. On it there are two invariant points which must be R, S. Consider a scroll Σ generated by lines cutting u, v, c_3; Σ will be of the fourth order, having v for a triple directrix and u for a simple one; it remains invariant in T_7. Now, consider, dually, a plane α and construct the torse $\varkappa_3$ determined by the five planes α, and having u for an axis. This torse remains invariant, and has two imaginary invariant planes (uR), (uS). The points of contact also remain invariant; hence, c_3 is self-dual and identical with $\varkappa_3$. Finally, regard c_3 as directrix of a linear complex; the surface Σ being self-dual and c_3 lying upon it, it appears that the generators of Σ all lie in the corresponding osculating planes of c_3; hence c_3 is an asymptotic line of the surface. By choosing another point on any generator, and passing a corresponding c_3 through it, all the asymptotic lines can be obtained. They all pass through the same points R, S, which are, therefore, pinch-points; they have the same osculating planes (Su), (Ru), which are the torsal planes, and, finally, all the same tangents, the limiting generators of the surface. Another set of scrolls similar to these can be obtained by interchanging u and v. *T_7 leaves ∞^3 quartic scrolls of the type* [3, 1] *invariant in such a way that the cubic asymptotic lines remain invariant.*

---

* See J. I. Hutchinson, "Note on the Tetrahedroid," in the Bulletin of the American Mathematical Society, vol. IV, p. 327, and "On a Special Form of a Quartic Surface," Annals of Mathematics, vol. XI, p. 158.

The general type of the equations of these scrolls, the invariant lines being chosen for $(x, y)(z, w)$ respectively are

$$\text{In } T_2: \quad x^2\phi_m(z, w) + y^2 f_m(z, w) = 0,$$

ϕ, f being binary quantics.

$$\text{In } T_3: \quad F(\lambda, \mu) = 0, \quad \lambda = x:y, \quad \mu = z:w.$$

In T_4 and T_7 the invariant scroll has the same form; usually the asymptotic lines on a [3, 1] scroll are of order 6,[*] here they are of order 3, and the pinch-points coincide in pairs, and are imaginary. Conversely, every [3, 1] having two double imaginary pinch-points has asymptotic lines of the third order.

Any [3, 1] scroll can be expressed in the form

$$zx(x^2 + 3cy^2) + wy(y^2 + 3cx^2) = 0.$$

The values of λ, which correspond to the pinch-points, are determined by the equation

$$3c^2 - 1 = c\left(\lambda^2 + \frac{1}{\lambda^2}\right),$$

which is to have two double imaginary roots. The admissible values of c are -1 and $-\frac{1}{3}$, which go into each other by interchanging x and y,

$$\Sigma \equiv zx(x^2 - 3y^2) + wy(y^2 - 3x^2) = 0.$$

The asymptotic lines are determined by this equation and

$$x\frac{\partial\Sigma}{\partial z} + xw\frac{\partial\Sigma}{\partial x} = 0.$$

---

[*] V. Snyder, "Asymptotic Lines on Ruled Surfaces," Bulletin Amer. Math. Society, vol. V, p. 348.

The projection of these curves on the plane $z=0$ gives

$$\left[(x^2-3y^2)\,x+\sqrt{\frac{\varkappa}{3}}\,(x^2+y^2)\,w\right]\left[(x^2-3y^2)\,x-\sqrt{\frac{\varkappa}{3}}\,(x^2+y^2)\,w\right]=0.$$

These curves all touch each other at the conjugate point $x=0$, $y=0$ projections of R and S. By changing the sign of the parameter $\varkappa$, all the lines become imaginary; hence, all the asymptotic lines of Σ belong to right-hand complexes. These two cubics make up a sextic, and their points of intersection with any generator are harmonic as to u and v; hence, they go into each other by T_3. The generators from every point of x, y are all real.

If five points $A_1 \ldots . A_5$ be chosen arbitrarily in space, no point A_6 can be found such that $T^6=1$, where $T\equiv(A_1A_2A_3A_4A_5A_6)$; for $T^2\equiv(A_1A_3A_5)(A_2A_4A_6)$ and $T^3\equiv(A_1A_4)(A_2A_5)(A_3A_6)$. T^2 is identical with T_4 and T^3 either with T_3 or T_2. It cannot be the latter, as the lines joining A_1A_4, A_2A_5 would then meet in a point O, which is contrary to the hypothesis that the points are chosen arbitrarily. Hence,

$$T^3\equiv T_3,\quad T^2\equiv T_4,$$

and only those configurations can remain invariant which are left so by both these transformations. In T_3 every point of u and v remains invariant, and every line joining corresponding points, as A_1A_4, cuts both. In T_4 the plane of each triple passes through v, and as A_1A_4 also cuts v, and similarly A_2A_5, A_3A_6 all of the points lie in a plane, contrary to hypothesis.

These planes all lie in the same pencil, and cut conics (containing the six points) from a fixed quadric surface, which, when the surface is projected into a hyperboloid of revolution, become parallel. T is now a rotation of 60° about the axis of this hyperboloid.

CORNELL UNIVERSITY, *April* 7, 1900.

On the Reduction of Hyperelliptic Integrals $(p=3)$ to Elliptic Integrals by Transformations of the Second and Third Degrees.

BY WILLIAM GILESPIE.

INTRODUCTION.

The principal subject of this paper is an application of cubic involution to the problem of the reduction to elliptic integrals, of hyperelliptic integrals of genus $p=3$ and of the first kind, by a rational transformation of the third degree.

It forms thus a continuation of Professor Bolza's researches on the cubic transformations of elliptic integrals and cubic reductions of hyperelliptics of genus $p=2$.*

It is also closely connected with the work of J. I. Hutchinson in his dissertation† where he applies the quadratic involution to a similar problem where $p=2$.

In Part I, the general case of cubic reduction is considered after the connection of the problem with the theory of cubic involution has been established (§1).

The invariant condition which must be satisfied, in the case of cubic reduction, by the octavic on whose square root the hyperelliptic integral depends, is obtained (§2), and the most general reducible integral is obtained in four general forms, the last two being independent of the cubic involution (§§6 and 7).

The main subject of Part II is the singular case of simultaneous reduction of the second and third degree; that is, where two hyperelliptic integrals having the same octavic are reducible, one by a quadratic transformation and the other by a transformation of the third degree (§3–§8). As an introduction to this part, the general case of quadratic reduction $(p=3)$ is considered (§ 1), and the special cases of more than one reducible integral with a given radical, are determined. A complete table of such cases is formed (§2).

---

* "Die cubische Involution und Dreitheilung," etc., and "Zur Reduction Hyperelliptischer Integrals," etc., Math. An., Bd. 50, pp. 68 and 314.

† "On the Reduction of Hyperelliptic Functions $(p=2)$," etc., Chicago, 1897.

In the last part, the involution is specialized to one containing two cubes. For this special form, the most general reducible integral is determined (§1) and the cases of simultaneous reduction of the second and third degree (§2). For one of the latter cases, in which the hyperelliptic curve can be reduced to the normal form

$$y^2 = x(1-x^6),$$

the reduction problem is studied by means of the Weierstrass-Picard theorem on the periods of reducible integrals (§3). This leads not only to a confirmation of the results previously obtained by algebraic methods, but also to a determination of all hyperelliptic integrals of the first kind belonging to the curve

$$y^2 = x(1-x^6),$$

which are reducible to elliptic integrals by rational transformations of any degree.

In the following review but the merest outline of the work can be given; further details of the work may be found in the dissertation on this subject (Chicago, 1899).

PART I.

§1.—*The Connection of the Problem with Cubic Involution.*

The method used by Jacobi* for the transformation of elliptic integrals, can be extended at once to the investigation of the reducibility of hyperelliptics, and furnishes the results:

In order that the hyperelliptic integral of the first kind and of genus $p=3$,

$$\int \frac{q(x_1 x_2)(xdx)}{\sqrt{R(x_1 x_2)}} \tag{1}$$

(where $q(x_1 x_2)$ is a quadratic and $R(x_1 x_2)$ an octavic quantic), *shall be reducible to the elliptic integral*

$$m \int \frac{(ydy)}{\sqrt{(y_1-\lambda_1 y_2)(y_1-\lambda_2 y_2)(y_1-\lambda_3 y_2)(y_1-\lambda_4 y_2)}}. \tag{2}$$

---

* "Fundamenta Nova," Werke, Bd. I, p. 55.

m being a constant by the rational transformation

$$\left.\begin{aligned} y_1 &= U, \\ y_2 &= V, \end{aligned}\right\} \tag{3}$$

where U and V are cubic quantics in $(x_1 x_2)$, *it is necessary and sufficient*, first, that R can be broken up in two cubic factors R_1 and R_2 and a quadratic $(x\varepsilon_1)(x\varepsilon_2)$ such that

$$\left.\begin{aligned} U - \lambda_1 V &= R_1(x_1 x_2), \\ U - \lambda_2 V &= R_2(x_1 x_2), \\ U - \lambda_3 V &= \quad (xd_1)^2(xe_1), \\ U - \lambda_4 V &= \quad (xd_2)^2(xe_2), \end{aligned}\right\} \tag{4}$$

or, what is the same, *R must be decomposable into three factors, two cubics which shall define a cubic involution and a quadratic whose roots are branch points of that involution;* and secondly, since $(ydy) = UdV - VdU$, which is proportional to the Jacobian of U and V whose roots are the double points of the involution, *the roots of the quadratic q must be the two double points not corresponding to the branch points in R.*

Thus a reducible integral must have the form

$$\int \frac{(xd_3)(xd_4)(xdx)}{\sqrt{(xe_1)(xe_2)(\lambda_1 f_1 + \lambda_2 f_2)(\mu_1 f_1 + \mu_2 f_2)}}, \tag{5}$$

where f_1 and f_2 are linear homogeneous functions of the cubics U and V and $(xd_i)^2(xe_i)$ for $i = 1, 2, 3, 4$, denote the branch triples of the involution $U + \lambda V$.

If an integral have this form, it is reducible.

§2.—*The Cubic Involution and the Invariant Condition for Reducibility.*

We collect in this paragraph those properties of the cubic involution of which we will make frequent use.

A cubic involution is defined by the equation

$$f_1 + \lambda f_2 = 0, \tag{6}$$

where λ is an arbitrary parameter and f_1 and f_2 two cubic quantics.

We will require the following combinants

$$\vartheta = (f_1 f_2)_1, \quad J = (f_1 f_2)_3, \quad H = (\vartheta\vartheta)_2, \quad i_\vartheta = (\vartheta\vartheta)_4, \quad j = (\vartheta H)_4, \tag{7}$$

Being given one point (y) of a triple, the other points are the roots of the affinity equation

$$3\vartheta_x^2 \vartheta_y^2 + \tfrac{1}{2} J (xy)^2 = 0. \tag{8}$$

The double points of the involution are the roots of ϑ, while the branch points are those of

$$3H_\vartheta + J\vartheta. \tag{9}$$

The first polar of

$$\Gamma = 2H_\vartheta + \tfrac{1}{3}J\vartheta, \tag{10}$$

with respect to an arbitrary parameter λ, is the cubic involution itself. For a given ϑ there exists, besides the involution spoken of above, another called the conjugate involution. For this involution all the statements made above hold good if we first change the sign of J.[*]

The condition which must be satisfied in the case of cubic reduction (§1) by the octavic R can be expressed thus:

In order that there shall exist a reducible hyperelliptic of the first kind belonging to R, R must be decomposable into the factors R_1, R_2 and N, where N is a quadratic dividing exactly the branch form $3H_\vartheta + J\vartheta$ for the involution determined by the cubic factors R_1 and R_2

This is equivalent to two algebraic relations between the roots of R. In accordance with the general theory of reduction,[†] it can be represented by the identical vanishing of a simultaneous biquadratic invariant of N and $3H + J\vartheta$.[‡]

§3.—*A Normal Form for the Reducible Integral.*

From the form of the reducible integral in §1, we see that the double points are associated in pairs, and for that reason we choose the following normal form used by Klein in his "Modulfunctionen" (p. 7):

Denoting the roots of ϑ by

$$x_1^{(i)} : x_2^{(i)} \quad i = 1, 2, 3, 4 \quad \text{and} \quad x_1^{(i)} x_2^{(k)} - x_2^{(i)} x_1^{(k)} \text{ by } (ik),$$

we have the invariants

$$A = (12)(34); \quad B = (13)(42); \quad C = (14)(23), \tag{11}$$

where

$$A + B + C = 0.$$

---

[*] We refer for these statements to Bolza's paper in the Annalen, Bd. 50, §1, where the literature of the subject is given.

[†] Koenigsberger, Crelle, 85.

[‡] Clebsch, "Binäre Formen," §27, p. 94.

Then, by a linear transformation of determinant 1, ϑ can be transformed into the form

$$(\vartheta) \quad x_1 x_2 (x_2 - x_1)(A x_1 + B x_2). \tag{12}$$

Here the pairing of the roots is put in evidence. The invariants may be expressed in the following manner:

$$A = 2i\sqrt{3}\, l_1 \bar{l}_1; \quad B = 2i\sqrt{3}\, l_2 \bar{l}_2; \quad C = 2i\sqrt{3}\, l_3 \bar{l}_3, \tag{13}$$

where the l's are the irrational invariants of the cubic involution used by Bolza* and are connected by the following relations:

$$l_1 + l_2 + l_3 = 0; \quad l_1 - l_2 = i\sqrt{3}\, \bar{l}_3; \quad l_2 - l_3 = i\sqrt{3}\, \bar{l}_1; \quad l_3 - l_1 = i\sqrt{3}\, \bar{l}_2. \tag{14}$$

Using the affinity equation and these values for the invariants, we find the following branch triples:

$$\begin{array}{lll}
 & \textit{double point.} & \textit{branch point.} \\
\text{I.} & x_1 = 0 & x_1 = i\sqrt{3}\,\dfrac{\bar{l}_2}{l_1}\, x_2; \\
\text{II.} & x_2 = 0 & x_1 = \dfrac{l_2}{i\sqrt{3}\,\bar{l}_1}\, x_2; \\
\text{III.} & x_1 = x_2 & x_1 = \dfrac{l_2}{l_1}\, x_2; \\
\text{IV.} & x_1 = -\dfrac{l_2 \bar{l}_2}{l_1 \bar{l}_1}\, x_2 & x_1 = -\dfrac{\bar{l}_2}{\bar{l}_1}\, x_2.
\end{array} \tag{15}$$

Then, using the first two double points as roots of q and the first two branch triples as U and V, we obtain the result:

Every hyperelliptic integral which is reducible by a transformation of the third degree, can be transformed by a linear transformation of the independent variable into the form

$$2i\sqrt{3} \int \frac{x_1 x_2 (x dx)}{\sqrt{\begin{array}{r}(l_1 x_1 - l_2 x_2)(\bar{l}_1 x_1 + \bar{l}_2 x_2)[\lambda_1 (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2) x_1^2 + \lambda_2 (i\sqrt{3}\, l_1 x_1 - l_2 x_2) x_2^2] \\ \times [\mu_1 (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2) x_1^2 + \mu_2 (i\sqrt{3}\,\bar{l}_1 x_1 - l_2 x_2) x_2^2]\end{array}}}, \tag{16}$$

which, by the transformation

$$y_1 = (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2) x_1^2, \quad y_2 = (i\sqrt{3}\,\bar{l}_1 x_1 - l_2 x_2) x_2^2,$$

---

[*] L. c., equation (77).

reduces to the elliptic integral

$$\int \frac{(ydy)}{\sqrt{(y_1+y_2)(l_1 l_1^3 y_1 - l_2 l_2^3 y_2)(\lambda_1 y_1 + \lambda_2 y_2)(\mu_1 y_1 + \mu_2 y_2)}}, \tag{17}$$

where $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ are arbitrary parameters and the l's are defined above; the integral thus depends on three independent non-homogeneous parameters, which agrees with the result in the end of §2.

§4.—*A Theorem on Cubic Involutions.*

The following lemma will help us to obtain the reducible integral in its most general form, independent of any normal form.

We have from Clebsch[*] immediately

$$\vartheta = \frac{2}{m_2 - m_3}(\chi^2 - \psi^2), \tag{18}$$

where ψ and χ are two of the quadratic factors of T_ϑ and the m's are roots of the cubic resolvent of ϑ.

We deduce:

(branch equation)
$$3H_\vartheta + J\vartheta = \frac{6}{m_2 - m_3}(l_3^2 \psi^2 - l_2^2 \chi^2), \tag{19}$$

$$\Gamma = 2H + \tfrac{1}{3} J\vartheta = \frac{4\bar{l}_1}{m_2 - m_3}(\bar{l}_2 \psi^2 - \bar{l}_3 \chi^2), \tag{20}$$

(affinity equation)
$$\frac{2}{m_2 - m_3}[\chi(x)\chi(y) - \psi(x)\psi(y)] + l_2 l_3 (xy)^2, \tag{21}$$

where
$$(xy) = x_1 y_2 - x_2 y_1.$$

If we form the product of the affinity equations for the two roots of the factor $l_3 \psi - l_2 \chi$ of the branch equation, we find it has the form

$$l_1^2 l_2^2 l_3^2 (\psi - \chi)^2.$$

Thus we obtain the following lemma:

In any cubic involution the Jacobian and the branch equation, when expressed in terms of the Clebsch functions ψ *and* χ *and the invariants* m's *and* l's, *assume the forms* $\frac{2}{m_2 - m_3}(\chi^2 - \psi^2)$ *and* $\frac{6}{m_2 - m_3}(l_3^2 \psi^2 - l_2^2 \chi^2)$ *respectively; the double points yielded by the factor* $(\psi - \chi)$ *of the Jacobian correspond to the branch points*

---

[*] "Binäre Formen," p. 159.

yielded by the factor $(l_3\psi - l_2\chi)$ *of the branch equation, and there is a similar correspondence between the roots of* $(\psi + \chi)$ *and* $(l_3\psi + l_2\chi)$.

For the conjugate involution, the factors $(\psi - \chi)$ and $(\bar{l}_3\psi - \bar{l}_2\chi)$ correspond, and also $(\psi + \chi)$ and $(\bar{l}_3\psi + \bar{l}_2\chi)$.

§5.—*First General Form of the Reducible Integral.*

This lemma of (§4) will be applied now to the form of the reducible integrals given in equation (5).

We may suppose

$$\left.\begin{aligned} &(\psi + \chi) = (xd_1)(xd_2)\,; && (\psi - \chi) = (xd_3)(xd_4)\,, \\ &(l_3\psi + l_2\chi = (xe_1)(xe_2)\;; && (l_3\psi - l_2\chi) = (xe_3)(xe_4)\;, \\ &(\bar{l}_3\psi + \bar{l}_2\chi = (x\bar{e}_1)(x\bar{e}_2)\;; && (\bar{l}_3\psi - \bar{l}_2\chi) = (\bar{x}e_3)(\bar{x}\varepsilon_4)\;. \end{aligned}\right\} \qquad (22)$$

This, however, involves an agreement in the choice of the homogeneous coordinates of the points $d_i\,e_i$ and $\bar{e}_i$.

Also, since Γ or $\frac{4i}{\sqrt{3\bar{l}_1}}(\bar{l}_2\psi^3 - \bar{l}_3\chi^3)$ has for its first polar with regard to an arbitrary parameter the cubic involution itself $(\Gamma_\infty^3\Gamma_\lambda)$, $\therefore$ the reducible integral (5) has the form

$$\int \frac{(\psi - \chi)(xdx)}{\sqrt{(l_3\psi + l_2\chi)\,\Gamma_x^3\Gamma_\lambda\,\Gamma_x'^3\Gamma_\mu'}} \qquad (23)$$

The transformation

$$y_1 = U, \quad y_2 = V$$

can, by a linear transformation of the elliptic integral, be thrown into the form $\Gamma_x^3\Gamma_y = 0$ or

$$y_1 = \frac{1}{4}\frac{\partial\Gamma}{\partial x_2}, \qquad y_2 = -\frac{1}{4}\frac{\partial\Gamma}{\partial x_1}. \qquad (24)$$

Let us now consider the elliptic integral.

Before reduction, the hyperelliptic integral has the form

$$\int \frac{\vartheta\,(xdx)}{\sqrt{(xd_1)^2\,(xe_1)(xd_2)^2\,(xe_2)\,\Gamma_x^3\Gamma_\lambda\,\Gamma_x'^3\,\Gamma_\mu'}}\,, \qquad (25)$$

and by the transformation (24)

$$\left.\begin{aligned} &(y\bar{e}_i) = \Gamma_x^3\Gamma_{\varepsilon_i} = \rho_i\,(xe_i)(xd_i)^2, \qquad i = 1,\ 2,\ 3,\ 4, && \text{[Bolza, l. c. (34)]} \\ &(y\lambda) = \Gamma_x^3\Gamma_\lambda, \quad \text{let } (y\lambda)(y\mu) = g\,(y), && \\ &\text{we also have} && \\ &(ydy) = \tfrac{8}{3}\,\Omega\vartheta\,(xdx), && \text{[Bolza, l. c. (38)]} \end{aligned}\right\} \qquad (26)$$

∴ after the transformation, our integral has the form

$$C\int\frac{(ydy)}{\sqrt{(\bar{l}_3\psi(y)+\bar{l}_2\chi(y))\,g(y)}}\,, \qquad (27$$

where C is a constant and g is an arbitrary quadratic having as roots the paran eters λ and μ.

Returning to the hyperelliptic integral (23), since

$$\Gamma^3_x\Gamma'^{3}_x\Gamma_\lambda\Gamma'_\mu=(\Gamma g)_2\Gamma-\tfrac{1}{2}H_\Gamma g, \qquad (28$$

it must have the form

$$\int\frac{(\psi-\chi)(xdx)}{\sqrt{(l_3\psi+l_2\chi)[(\Gamma g)_2\Gamma-\frac{1}{2}H_\Gamma g)]}}\,. \qquad (2$$

Equations (24) and (27) give us the transformation and the elliptic integra but the constant factor must be obtained.

To find this factor, we must collect those from the following equations:

$$(ydy)=\tfrac{3}{2}\Omega\,.\,\vartheta\,.\,(xdx), \quad \text{see (26)}, \qquad (3$$

$$\Gamma^3_x\Gamma_{\bar\epsilon_1}\Gamma'^{3}_x\Gamma'_{\bar\epsilon_2}=\frac{\Omega}{l_1\bar{l}_2}(l_3\psi+l_2\chi)(\psi+\chi)^2, \qquad (3$$

where $\sqrt{\Omega}=2\bar{l}_1\bar{l}_2\bar{l}_3$ and its conjugate $\sqrt{\Omega}=2l_1l_2l_3$., [Bolza, l. c., §7], (3

and also the factor from

$$\vartheta=\frac{2}{i\sqrt{3}l_1\bar{l}_1}(\psi^2-\chi^2), \quad \text{see (18)}, \qquad (3$$

we have thus the theorem:

The hyperelliptic integral

$$\frac{\sqrt{3\Omega}}{i\sqrt{l_1\bar{l}_1}}\int\frac{(\psi-\chi)(xdx)}{\sqrt{(l_3\psi+l_2\chi)[(g\Gamma)_2\Gamma-\frac{1}{2}H_\Gamma g]}}\,, \qquad (3$$

by the transformation $\rho y_1=\frac{1}{4}\frac{\partial\Gamma}{\partial x_2}$, $\rho y_2=-\frac{1}{4}\frac{\partial\Gamma}{\partial x_1}$ *reduces to the elliptic integral*

$$\int\frac{(ydy)}{\sqrt{(\bar{l}_3\psi(y)+\bar{l}_2\chi(y))\,g(y)}}\,. \qquad (3$$

§6.—*A Second General Form of the Reducible Integral.*

It is possible to express these integrals and the transformation in terms of the biquadratic Γ, so that, knowing Γ, we can find the integral.

We know that $$T_\theta = 2\phi\psi\chi, \qquad \text{[Clebsch, l. c., p. 159].}$$

Let now $$T_\Gamma = 2\Phi\Psi\mathrm{X}, \tag{36}$$

and let n_1, n_2, n_3 be the roots of the resolvent of Γ. Then it is found that

$$\phi = \frac{1}{\sqrt{2n_1}}\Phi, \quad \psi = \frac{1}{\sqrt{2n_2}}\Psi, \quad \chi = \frac{1}{\sqrt{2n_3}}\mathrm{X}, \tag{37}$$

and using the relations* that exist between the l's, n's and Ω, we obtain the reducible integral (34) in a new form, whence the theorem:

If Γ *be any biquadratic quantic and* Φ, Ψ *and* X *the three quadratic factors of* T_Γ, n_1, n_2 *and* n_3, *the roots of the resolvent of* Γ *and* g *an arbitrary quadratic, then the most general reducible hyperelliptic integral has the form*

$$\frac{3}{(n_2 - n_3)^{\frac{1}{4}}}\int\frac{(\sqrt{n_3}\,\Psi - \sqrt{n_2}\,\mathrm{X})(xdx)}{\sqrt{[\sqrt{n_3}\,(n_1 - n_2))\,\Psi + \sqrt{n_2}\,(n_3 - n_1)\,\mathrm{X}][(\Gamma g)_2\,\Gamma - \frac{1}{2}H_\Gamma\, g]}} \tag{38}$$

and reduces to the elliptic integral

$$\int\frac{(ydy)}{\sqrt{[\sqrt{n_3^3}\,\Psi(y) + \sqrt{n_2^3}\,\mathrm{X}(y)]\,g(y)}} \text{ by the transformation } \begin{cases} y_1 = \frac{1}{4}\dfrac{\partial\Gamma}{\partial x_2}, \\ y_2 = -\frac{1}{4}\dfrac{\partial\Gamma}{\partial x_1} \end{cases}$$

This form is completely independent of the cubic involution.

§7.—*A Third General Form of the Reducible Integral.*

Finally, the integral can be expressed in terms of three quadratics ξ, η and ζ, defined in the following way:

$$\left.\begin{aligned} \Phi &= \frac{i\sqrt{3}}{2}\,\Omega^{\frac{1}{4}}\sqrt{l_2 l_3}\,\xi, \\ \Psi &= \frac{i\sqrt{3}}{2}\,\Omega^{\frac{1}{4}}\sqrt{l_3 l_1}\,\eta, \\ \mathrm{X} &= \frac{i\sqrt{3}}{2}\,\Omega^{\frac{1}{4}}\sqrt{l_1 l_2}\,\zeta, \end{aligned}\right\} \tag{39}$$

---

* Bolza, l. c., (85), (86).

Since the functions Φ, Ψ and X are mutually harmonic, it follows that ξ, η and ζ are also, and the discriminant of each can be shown to be equal to 2.

Substituting in these values in (38), we obtain the theorem:

If ξ, η, ζ be three quadratic quantics mutually harmonic and so normalized that the discriminant of each is 2, and if also l_1, l_2, l_3, $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$ be any quantities which satisfy the four relations

$$l_1+l_2+l_3=0,\quad l_1-l_2=i\sqrt{3}\,\bar{l}_3,\quad l_2-l_3\,i\sqrt{3}\,\bar{l}_1,\quad l_3-l_1=i\sqrt{3}\,\bar{l}_2,$$

and if g be an arbitrary quadratic, then the most general reducible hyperelliptic integral has the form

$$i\sqrt{3}\int\frac{[\sqrt{l_3\bar{l}_3}\,\eta-\sqrt{\bar{l}_3 l_2}\,\zeta](xdx)}{\sqrt{(\sqrt{\bar{l}_3^3 l_3}\,\eta+\sqrt{l_2\bar{l}_2}\,\zeta)[(l_3B\eta-l_2C\zeta)(l_3\eta^2-l_2\zeta^2)-g(\bar{l}_3^2\eta^2+\bar{l}_2^2\zeta^2)]}}, \tag{40}$$

(where B and C are constants equal to $(\eta g)_2$ and $(\zeta g)_2$ respectively), *and reduces to the elliptic integral*

$$\int\frac{(ydy)}{\sqrt{(\sqrt{\bar{l}_3^3 l_3}\,\eta(y)+\sqrt{\bar{l}_2^3 l_2}\,\zeta(y))g(y)}}$$

by the cubic transformation

$$l_3\eta_x'^2\eta_x\eta_y-l_2\zeta_x'^2\zeta_x\zeta_y=0,$$

where $\eta_x^2=\eta_x'^2=\eta(x)$ and $\zeta(x)=\zeta_x^2=\zeta_x'$.

This form is completely independent of the involution.

PART II.

§1.—*General Case of Reduction of the Second Degree.*

As an introduction to this part, we consider the general form of a hyperelliptic integral of the first kind and of genus 3, which is reducible by a rational transformation of the second degree.

Using again the method of Jacobi just as in Part I, §1, we deduce the theorem:

In order that the hyperelliptic integral

$$\int\frac{q(x_1x_2)(xdx)}{\sqrt{R(x_1x_2)}} \tag{41}$$

(where q is a quadratic and R an octavic quantic), *shall be reducible to the elliptic integral*

$$m\int\frac{ydy}{\sqrt{(y_1-\lambda_1 y_2)(y_1-\lambda_2 y_2)(y_1-\lambda_3 y_2)(y_1-\lambda_4 y_2)}},$$

m being a constant, *by a rational transformation*

$$y_1 = U$$
$$y_2 = V'$$

where U and V are quadratic quantics in $(x_1 x_2)$, *it is necessary and sufficient that R shall be decomposable into four factors belonging to the same quadratic involution, and that the quantic q shall have as roots the double points of that involution.*

By a parametric representation of the roots of R as points on a conic section, we have a geometrical statement for these conditions. The points of R must be so situated that we can pair them in such a way that the four joins shall pass through a common point, and the tangents to the conic from this point shall touch in the points which represent the roots of q.

§2.—*Singular Cases of Reduction of the Second Degree.*

From the conditions just determined, we conclude that in general for a given octavic R, which fulfills the required condition of §1, there exists but one reducible integral. Some octavics, however, are decomposable in the required way in more ways than one. In such cases, more than one reducible integral will exist with the same octavic R.

The singular cases can be determined by the method used by J. I. Hutchinson in his dissertation (Chicago, 1897). First determine all octavics invariant under the finite linear substitution groups, then, for each octavic, pick out all the transformations for period 2 which interchange the eight roots in pairs.

Having all these transformations, we can form all the reducible integrals. The stationary or double points of each transformation are the roots of q, the numerator of the integral, and the R is in each case the invariant octavic. We may use for the reducing transformation

$$y_1 = (xd_1)^2, \quad y_2 = (xd_2)^2,$$

where d_1 and d_2 are the stationary points of the transformation.

The following table contains an enumeration of all such octavics, the groups under which each is invariant, the stationary points and the number of distinct reducible hyperelliptic integrals which can be obtained for each octavic:

	R.	Group.	Stationary points.	Number of reducible integrals.
I (a)	$x_1^8 + 4ax_1^6 x_2^2 + 6bx_1^4 x_2^4 + 4cx_1^2 x_2^6 + dx_2^8$	cyclic $n = 2$	$x_1 x_2 = 0$	1
I (b)	$x_1 x_2 (x_1^6 + ax_1^4 x_2^2 + bx_1^2 x_2^4 + cx_2^6)$	cyclic $n = 2$		none
II	$x_1 (x_1^7 + x_2^7)$	cyclic $n = 7$		none
III (a)	$x_1^8 + ax_1^6 x_2^2 + bx_1^4 x_2^4 + ax_1^2 x_2^6 + x_2^8$	dihedron $n = 2$	$x_1 x_2 = 0$; $x_1^2 = x_2^2$; $x_1^2 = -x_2^2$	3
III (b)	$x_1 x_2 (x_1^2 + x_2^2)(x_1^4 + ax_1^2 x_2^2 + x_2^4)$	dihedron $n = 2$	$x_1^2 = x_2^2$	1
IV	$x_1 x_2 (x_1^6 + ax_1^3 x_2^3 + x_2^6)$	dihedron $n = 3$	$x_1^2 = \varepsilon^{\frac{2n}{3}\pi i} x_2^2$; $(n = 0.\ 1.\ 2)$	3
V	$x_1^8 + ax_1^4 x_2^4 + x_2^8)$	dihedron $n = 4$	$x_1 x_2 = 0$; $x_1^2 = \varepsilon^{\frac{n\pi i}{2}} x_2^2$; $(n = 0.\ 1.\ 2.\ 3)$	5
VI	$x_1 x_2 (x_1^6 + x_2^6)$	dihedron $n = 6$	$x_1^2 = \varepsilon^{\frac{2n+1}{6}\pi i} x_2^2$; $(n = 0.\ 1.\ 2)$	3
VII	$x_1^8 - x_2^8$	dihedron $n = 8$	$x_1 x_2 = 0$; $x_1^2 = \varepsilon^{\frac{2n+1}{4}\pi i} x_2^2$; $(n = 0.\ 1.\ 2.\ 3)$	5
VIII	$x_1^8 + 14x_1^4 x_2^4 + x_2^8$	octahedron	$x_1 x_2 = 0$; $x_1^2 = x_2^2$; $x_1^2 = -x_2^2$ $x_1^2 = ix_2^2$; $x_1^2 = -ix_2^2$	9

§3.—*Cases of Simultaneous Reduction of the Second and Third Degree.*

We will now turn our attention to the problem of the determination of all the cases where there exist two hyperelliptic integrals of the first kind and genus 3, which have the same $R(x_1 x_2)$ and where one is reducible by a transformation of the second degree, while the other is reducible by one of the third.

We consider this from a geometrical standpoint.

By means of a parametric representation of a point on a conic section (C_2), Weyr gave the following geometric interpretation of a cubic involution:

The points of any triple of the involution define a triangle inscribed in the conic (C_2) and circumscribed about a conic (I_2) called the Involution Conic.

The points of intersection of (C_2) and (I_2) are the branch points of the involution, and the double points are the points where the common tangents to (C_2) and (I_1) touch (C_2).

If we choose for the triangle of reference the self-conjugate triangle of (C_2) and (I_2), the conics assume the equations*

$$\left.\begin{aligned}(C_2)\quad & x^2+y^2+z^2=0,\\ (I_2)\quad & \frac{x^2}{l_1^2}+\frac{y^2}{l_2^2}+\frac{z^2}{l_3^2}=0,\end{aligned}\right\} \tag{43}$$

where the l's are the invariants defined in Part I, (13).

The coordinates of the double points are

$$\rho x=\sqrt{l_1\bar{l}_1},\quad \rho y=\pm\sqrt{l_2\bar{l}_2},\quad \rho z=\pm\sqrt{l_3\bar{l}_3}, \tag{44}$$

and those of the branch points are

$$\rho x=l_1\sqrt{l_1\bar{l}_1},\quad \rho y=\pm l_2\sqrt{l_2\bar{l}_2},\quad \rho z=\pm l_3\sqrt{l_3\bar{l}_3}. \tag{45}$$

If, now, $R(x_1 x_2)$ is to permit both quadratic and cubic reduction of the integral, the eight points denoted by $R(x_1 x_2)$ must be arranged on (C_2) as follows:

Since cubic reduction is possible, three of the points (A, B and C) of R must define a triangle inscribed in (C_2) and circumscribed about (I_2), three more (A', B', C') must define another such triangle, while the other two points, (D and E) of R must be points of intersection of (C_2) and (I_2).

Again, since quadratic reduction is also possible, it must be possible to pair these eight points so that the joins of the four pairs shall pass through a common point O.

---

*Bolza, l. c., p. 90.

We obtain in all the following essentially different ways of pairing the eight points:

$$\left.\begin{array}{lllll} \text{I} & AA'; & BB'; & CC'; & DE. \\ \text{II} & AB; & A'B'; & CC'; & DE. \\ \text{III} & AB; & A'B'; & CD; & C'E. \\ \text{IV} & AA'; & B'C'; & BD; & CE. \\ \text{V} & AA'; & BB'; & CD; & C'E. \end{array}\right\} \tag{46}$$

§4.—*Case I, AA', BB', CC' and DE intersect in O.*

For this case the following theorems are proved:

I. If in a cubic involution two triples are such that they form, by taking one point, from each, three couples in quadratic involution, the branch points of the cubic involution form two couples in the quadratic involution. This may be stated thus: If the points of any two triples of a cubic involution are in perspective, the centre of perspectivity is the vertex of the self-conjugate triangle of the conic (C_2) and the involution conic (I_2).

II. Conversely, if we project a triple of the involution from a vertex of the self-conjugate triangle upon the normal conic (C_2), we obtain another triple of the involution.

Thus, being given ϑ and the sign of J, which are necessary for the construction of the conics (C_2) and (I_2), we may take for O any vertex of the self-conjugate triangle, we then choose for A any point on (C_2), the two tangents from A to (I_2) determines B and C and projecting A, B and C from O, we obtain other triple A', B' and C'. This is the desired arrangement.

Notice A is perfectly arbitrarily chosen on (C_2) and O is one of three points; we have thus an infinitude of solutions. In all other cases we have a finite number.

In order that the most general reducible integral, §7, P I, shall specialize to the present case of simultaneous reduction by a cubic and by a quadratic transformation, it is necessary and sufficient that the quadratic $g(x)$ shall be chosen apolar to the quadratic ξ.

§5.—*Case II, AB, A′B′, CC′ and DE intersect in O.*

Take the point O arbitrarily in the plane, and from it draw the two tangents to (I_2), touching in K and K' and cutting (C_2) in A and B and A' and B' respectively.

The line KK' is the polar of O with regard to (I_2), but from a relation* between the coordinates of K and C, the third point of the triple ABC and between K' and C', we learn that the line CC' is the polar of the point O with regard to a conic (B_2);

$$(B_2) \qquad \frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0. \tag{47}$$

But, since CC' is to pass through O, O must be on (B_2); since, also, DE is to pass through O, O must be a point of intersection of (B_2) and DE.

We may take any two branch points for DE. Two of the allowable positions for O are

$$\rho x = l_1 \sqrt{l_1 \bar{l}_1}, \quad \rho y = l_2 \sqrt{l_2 \bar{l}_2}, \quad \rho z = \pm i \sqrt{l_3 \bar{l}_3 (l_1 l_2 + l_3^2)}. \tag{48}$$

There will be twelve allowable positions for O.

§6.—*Properties of the Conic* (B_2).

This conic (B_2) has some peculiar properties, and is closely connected with the involution—

I. The conic (B_2) passes through the double points of the involution on C_2.

II. The conic (B_2) is apolar† to (I_2).

This, together with I, determines (B_2) completely; (B_2) is also apolar to (C_2).

III. A triangle inscribed in (C_2) and circumscribed about (I_2) is self-conjugate with regard to (B_2).

This is the geometrical meaning of II above.

From III follows at once a theorem which we reached otherwise in the last paragraph.

---

* O. Bolza, l. c. 60 (a).

† Meyer, "Apolarität und Rationale Curven," p. 84.

Draw from any point O two tangents to (I_1), cutting (C_2) in AB and $A'B'$ respectively; the line, joining C and C' which complete the triples of which AB and $A'B'$ are point pairs, is the polar of O with respect to the conic (B_2).

IV. The points in which the common tangents to (B_2) and (C_2) touch (C_2) are the four points $\overline{\Gamma} = 0$.

§7.—*Case III, AB, $A'B'$, CD, $C'E$ intersect in a point.*

Take arbitrarily the point S in the plane. Join S to E and D, two of the points of intersection of (C_2) and (I_2); produce these lines to cut the conic (C_2) again in C' and C respectively; the lines AB and $A'B'$ are thus defined, being the polars of C and C' with regard to the conic (B_2).

If AB and $A'B'$ meet in O, then, in order to fulfill the conditions of our problem, O and S must coincide.

If we use the branch points

$$(D \text{ and } E) \quad \left\{\begin{aligned} \rho x &= l_1\sqrt{l_1\bar{l}_1} = a, \\ \rho y &= l_2\sqrt{l_2\bar{l}_2} = b, \\ \rho z &= \pm\, l_3\sqrt{l_3\bar{l}_3} = \pm\, c. \end{aligned}\right\} \tag{49}$$

and impose the condition that O and S are coincident, we find the only permissible positions for O are the points of intersection of the conics:

$$\left.\begin{aligned} xz\,(l_1 - l_3)(bx - ay) &= l_1 b\,(x^2 + y^2 + z^2), \\ yz\,(l_3 - l_2)(bx - ay) &= l_2 a\,(x^2 + y^2 + z^2). \end{aligned}\right\} \tag{50}$$

Two of these points are the points D and E themselves, and the other two lie one on each of the tangents to the conic (I_2) at the points D and E.

The integrals arising from this case are, however, degenerate, as the radical is over an expression of the fourth degree only, since in each case one of the triples ABC or $A'B'C'$ is a branch triple including a branch point already in R.

§8.—*Cases IV and V.*

These two cases lead to more complicated results. The positions permissible for O are the points of intersection of curves of higher degree than the second, and as no very interesting theorems appear in the work, the discussion of these cases will be omitted.

PART III.

§1.—*Involutions containing Two Cubes.*

Up to this point we have supposed the involution to be perfectly general, but let us now consider the special case, where the cubic involution contains two cubes.

As Clebsch has shown,* every such involution may be expressed in the form

$$\lambda_1 f + \lambda_2 Q = 0, \tag{51}$$

where $Q = (f\Delta)$ and $\Delta = (ff)_2$.

Then using the notation of Clebsch, we find

$$\left.\begin{array}{l} \vartheta = (fQ) = -\tfrac{1}{2}\Delta^2; \quad J = (fQ)_3 = R, \\ H_\vartheta = \tfrac{1}{12} R\Delta^2; \quad \Gamma \equiv 0; \quad 3H + J\theta = -\tfrac{1}{2} R\Delta^2. \end{array}\right\} \tag{52}$$

Hence, by applying method of Part I, §1, we find:

The most general reducible hyperelliptic integral corresponding to this special form of the involution has the form

$$\int \frac{\Delta\,(xdx)}{\sqrt{\Delta\,(\lambda_1 f + \lambda_2 Q)(\mu_1 f + \mu_2 Q)}}. \tag{53}$$

The conditions necessary and sufficient for such a special form of the involution are

$$J \neq 0, \quad \Gamma \equiv 0.$$

But we can express the involution (51) in the form

$$\lambda_1' x_1^3 + \lambda_2' x_2^3 = 0. \tag{54}$$

Then $\vartheta = x_1^2 x_2^2$ and $3H + J\vartheta = \tfrac{1}{2} x_1^2 x_2^2$, and it follows the hyperelliptic integral

$$\int \frac{x_1 x_2\,(xdx)}{\sqrt{x_1 x_2 (\lambda_1' x_1^3 + \lambda_2' x_2^3)(\mu_1' x_1^3 + \mu_2' x_2^3)}} \tag{55}$$

---

[*] L. c., §89.

is reducible to the elliptic integral

$$\int\frac{(ydy)}{\sqrt{y_1y_2(\lambda_1'y_1+\lambda_2'y_2)(\mu_1'y_1+\mu_2'y_2)}}$$

by the transformation

$$y_1=x_1^3,$$
$$y_2=x_2^3.$$

§2.—*Cases of Simultaneous Reduction of the Second and Third Degree.*

Is it possible that the octavic which occurred in the integral (53)

$$\Delta\,(\lambda_1f+\lambda_2\,Q)(\mu_1f+\mu_2\,Q)$$

or its normal form

$$x_1x_2\,(\lambda_1'x_1^3+\lambda_2'x_2^3)(\mu_1'x_1^3+\mu_2'x_2^3) \tag{56}$$

can be decomposed in such a way as to yield also a quadratic reduction of the integral?

If this is possible, then the octavic (56) must coincide with some one of the invariant octavics found in the table, Part II, §2.

If $\lambda_1':\lambda_2'=\mu_2':-\mu_1'$, then the octavic is invariant under the dihedron group $(n=3)$, and, by making use of our table, this octavic yields us three reducible integrals. If, in particular, $\lambda_1':\lambda_2'=\mu_2':-\mu_1'=1$, the octavic assumes the form

$$x_1x_2\,(x_1^6-x_2^6), \tag{57}$$

which is invariant under the dihedron group $n=6$, and we obtain from the table, Part II, §2, the following three reducible integrals:

$$\int\frac{(x_1^3-\varepsilon^{\frac{2n+1}{3}\pi i}x_2^3)(xdx)}{\sqrt{x_1x_2\,(x_1^6-x_2^6)}},\qquad\text{for } n=0,\ 1,\ 2. \tag{58}$$

The octavic $\Delta\,(\lambda_1f+\lambda_2\,Q)(\mu_1f+\mu_2\,Q)$ will be invariant under the dihedron group $n=6$ if

$$\mu_1:\mu_2=-\frac{R}{2}\lambda_2:\lambda_1,$$

where R has the meaning given in (52).

If, now, we choose our notation so that

$$\lambda_1 f + \lambda_2 Q = f_{\lambda_1 \lambda_2} = f \text{ and } \mu_1 f + \mu_2 Q = Q_{\mu_1 \mu_2} = Q,$$

we have the reducible integrals

$$\int \frac{a_x^2 a_{\alpha_i}(xdx)}{\sqrt{\Delta . Q . f}}, \tag{59}$$

where $$a_x^2 = f \text{ and } Q = (x\alpha_1)(x\alpha_2)(x\alpha_3)$$

and the theorem:

For a given octavic $\Delta f Q$ we have altogether four reducible integrals, one (53), whose numerator is Δ, by a transformation of the third degree, and three others whose numerators are the first polars of f with regard to the different roots of Q, by transformations of the second degree.

§3.—*Integrals Treated by Method of Periods.*

In the last part of the paper, the reduction problem is studied for a special hyperelliptic curve

$$y^2 = x(1 - x^6) \tag{60}$$

by means of the Weierstrass-Picard theorem on the periods of reducible integrals.

A system of three linearly independent integrals of the first kind, with y for denominators is chosen, a Riemann surface is constructed and a canonical system of cuts made.

The moduli of periodicity of the three integrals at the various cuts are tabulated, and by repeated applications of the Weierstrass-Picard theorem certain reducible integrals are found which all have the denominator $y = \sqrt{x(1-x^6)}$. Among these there are three reducible by quadratic transformation, and are exactly the three given in equation (58), where they were obtained in an entirely different manner. Finally, the following theorem is obtained:

Every integral of the form

$$(a_1 + b_1 i)\frac{\Gamma\frac{3}{4}}{\Gamma\frac{1}{4}}\ \frac{1}{\sqrt{\pi}}\int\frac{x \,.\, dx}{\sqrt{x(1-x^6)}} + (a_2 + b_2 i)\frac{\sqrt{2\pi}(1-i)}{\Gamma\frac{5}{12}\Gamma\frac{1}{12}}\int\frac{(1-x^2)\,dx}{\sqrt{x(1-x^6)}}$$

$$+ (a_3 + b_3 i)\frac{\sqrt{6\pi}(1+i)}{\Gamma\frac{1}{12}\Gamma\frac{5}{12}}\, e^{\frac{2\pi i}{3}}\int\frac{(x^2 - e^{\frac{5\pi i}{3}})\,dx}{\sqrt{x(1-x^6)}}$$

(where a_i and b_i are rational), *is reducible to an elliptic integral, and there are no other hyperelliptic integrals of the first kind belonging to the equation*

$$y^2 = x(1-x^6)$$

which are reducible by a transformation of any degree.

The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions.[*]

By Eliakim Hastings Moore, *of Chicago.*

Introduction.

The binary n-ic form has as an absolute irrational invariant the cross-ratio of any four of its roots. These cross-ratios are expressible rationally in terms of any $n-3$ independent ones. If any particular system of $n-3$ independent ratios be associated with a particular order of the n roots, by varying the order of the n roots, we shall have in all $n!$ conjugate systems; these systems are expressible rationally in terms of the original system, and exactly so in terms of any system of the set. Hence arises, to speak geometrically, a group of $n!$ Cremona transformations in flat space of $n-3$ dimensions. The various Cremona groups so obtained from the various initial systems are Cremona transformations of one another. In this paper I study more closely one of the simplest of such Cremona groups.

§1.

Definition of the cross-ratio group $G_{n!}$ of Cremona transformations $F^{(a)}$.

I recall certain fundamental properties of the cross-ratio rational function $[xyzu]$ of the four independent variables $xyzu$:

$$[xyzu]=\frac{(x-z)(y-u)}{(y-z)(x-u)}.$$

---

[*] This paper, with slight modifications, was read before the American Mathematical Society at the Buffalo meeting of the summer of 1896.

The papers of G. Kohn, "Ueber die Erweiterung eines Grundbegriffes der Geometrie der Lage" (Mathematische Annalen, vol. 46, p. 285, 1895), and "Die homogenen Coordinaten als Wurfcoordinaten" (Wiener Sitzungsberichte, vol. 104, p. 1167, 1895), have certain connections of content with the present (independent) paper.

(*a*) $$[xyzu] \,.\, [wxzu] = [wyzu].$$

(*b*) $$[xyzu] = [yxuz] = [zuxy] = [uzyx].$$

(*c*) $$[xyzx] = \infty, \quad [xyzy] = 0, \quad [xyzz] = 1; \; [\infty 0 1 u] = u.$$

(*d*) $$[xyzu] \,.\, [yxzu] = 1, \quad [xyzu] + [xzyu] = 1.$$

(*e*) The four quantities $xyzu$ taken in all $4! = 24$ orders give rise to six, and only six, different cross-ratios, and these are expressible as linear fractional functions of (any) one of them, viz.:

$$\begin{aligned}
[xyzu] &= \lambda && = \lambda_0(\lambda), & [xzyu] &= 1 - \lambda && = \lambda_2(\lambda), \\
[yzxu] &= \frac{\lambda - 1}{\lambda} && = \lambda_1(\lambda), & [yxzu] &= \frac{1}{\lambda} && = \lambda_4(\lambda), \\
[zxyu] &= \frac{1}{1 - \lambda} && = \lambda_3(\lambda), & [zyxu] &= \frac{\lambda}{\lambda - 1} && = \lambda_5(\lambda).
\end{aligned}$$

(*f*) The cross-ratio $[xyzu]$ is invariant under cogredient linear fractional transformation

$$v' = \frac{av + b}{cv + d} \qquad (ad - bc \neq 0)$$

of its variables x, y, z, u:

$$[xyzu] = [x'\, y'\, z'\, u].$$

(*g*) x, y, z being unequal, by the transformation

$$v' = \frac{(x - z)(y - v)}{(y - z)(x - v)} = [xyzv],$$
$$(ad - bc = -(x - y)(x - z)(y - z) \neq 0),$$

which throws $v = x, y, z$ to $v' = \infty, 0, 1$, we have

$$[xyzu] = [x'\, y'\, z'\, u'] = [\infty 0 1 u'] = u'.$$

We take now n independent variables

$$z_i \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

and consider the $n(n-1)(n-2)(n-3)$ cross-ratios

$$r_{ijkl} = [z_i z_j z_k z_l], \qquad (i, j, k, l = 1, 2, \ldots, n). \quad (2)$$

Of the n ratios

$$r_i = [z_n z_{n-1} z_{n-2} z_i], \qquad (i = 1, 2, \ldots, n), \qquad (3)$$

the three r_n, r_{n-1}, r_{n-2} have the numerical values ∞, 0, 1 respectively, while the $n-3$ remaining $r_i\,(i=1, 2, \dots, n-3)$ are obviously independent and form, we say, a *fundamental system*

$$R = (r_1, r_2, \dots, r_{n-3}) \tag{4}$$

of $n-3$ cross-ratios for the rational expression (in accordance with the remarks a, b, d) of all the cross-ratios r_{ijkl}. Indeed, by applying the transformation $v' = [z_n z_{n-1} z_{n-2} v]$ to the n variables $v = z_i$, we have (by g, f), since $r_i = z_i'$,

$$r_{ijkl} = [r_i r_j r_k r_l]. \tag{5}$$

Similarly every order a of the n variables $z_1 \dots z_n$,

$$a = (z_{a_1} z_{a_2} \dots z_{a_n}), \tag{6}$$

gives rise to a corresponding fundamental system

$$R^{(a)} = (r_1^{(a)}, r_2^{(a)}, \dots, r_{n-3}^{(a)}), \quad r_i^{(a)} = [z_{a_n} z_{a_{n-1}} z_{a_{n-2}} z_{a_i}], \quad (i = 1, 2, \dots, n-3). \tag{7}$$

The two fundamental systems R, $R^{(a)}$ are, of course, expressible each in terms of the other. We write the system of $n-3$ equations

$$r_i^{(a)} = [r_{a_n} r_{a_{n-1}} r_{a_{n-2}} r_{a_i}] = f_i^{(a)}(r_1, r_2, \dots, r_{n-3}) \quad (i = 1, 2, \dots, n-3) \tag{8}$$

more compactly

$$(r_1^{(a)}, r_2^{(a)}, \dots, r_{n-3}^{(a)}) = (f_1^{(a)}, f_2^{(a)}, \dots, f_{n-3}^{(a)})(r_1, r_2, \dots, r_{n-3}), \tag{8'}$$

or

$$R^{(a)} = F^{(a)} R, \tag{8''}$$

where, on the right, $F^{(a)} = (f_1^{(a)}, f_2^{(a)}, \dots, f_{n-3}^{(a)})$ is a system of $n-3$ functional operators each on $n-3$ arguments.

Now, in a flat space R_{n-3} of $n-3$ dimensions, a point R being determined by the $n-3$ non-homogeneous point-coordinates r_i,

$$R = (r_1, r_2, \dots, r_{n-3}), \tag{9}$$

we determine by

$$R' = F^{(a)} R, \tag{10}$$

a 1, 1 rational point-transformation; that is, a *Cremona transformation $F^{(a)}$ of the flat space R_{n-3}*. The *order* of this transformation $F^{(a)}$, that is, the order of the $(n-4)$-way locus-spread in the flat space R_{n-3} of points R which corresponds to the general linear (flat) spread R_{n-4} in the flat space R_{n-3} of points R', is

$$1 \text{ if } a_n = n, \quad n-3 \text{ if } a_n \neq n.$$

For, indeed, the transformation $F^{(a)}$ (8, 8')

$$r'_1, \ldots, r'_i, \ldots, r'_{n-3}$$
$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_{n-1}} - r_{a_{n-2}}} \left(\frac{r_{a_{n-1}} - r_{a_1}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_n} - r_{a_{n-3}}} \right) \quad (11)$$

is for the four cases $n = a_n,\ a_{n-1},\ a_{n-2},\ a_j\ (j \leqq n-3)$, when we recall that $r_n = \infty$:

$a_n = n)$ $\quad r'_1, \ldots, r'_i, \ldots, r'_{n-3}$

$$= \frac{r_{a_{n-1}} - r_{a_1}}{r_{a_{n-1}} - r_{a_{n-2}}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_{n-1}} - r_{a_{n-2}}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_{n-1}} - r_{a_{n-2}}}. \quad (11_1)$$

$a_{n-1} = n)$ $\quad r'_1, \ldots, r'_i, \ldots, r'_{n-3}$

$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_{n-3}}}. \quad (11_2)$$

$a_{n-2} = n)$ $\quad r'_1, \ldots, r'_i, \ldots, n_{n-3}$

$$= \frac{r_{a_{n-1}} - r_{a_1}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_n} - r_{a_{n-3}}}. \quad (11_3)$$

$a_j \quad = n)$ $\quad \ldots, r'_i, \ldots, r'_j, \ldots$

$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_{n-1}} - r_{a_{n-2}}} \left(\ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, 1, \ldots \right) \quad (11_4)$$
$$(i \neq j,\ i = 1, 2, \ldots, n-3).$$

The *literal substitution* α

$$\alpha = \begin{pmatrix} z_1, & \ldots, & z_i, & \ldots, & z_n \\ z_{a_1}, & \ldots, & z_{a_i}, & \ldots, & z_{a_n} \end{pmatrix} = (z_i z_{a_i}) \quad (12)$$

changes the initial order o to the order a. There is a *composition of substitutions*

$$\alpha\beta = \gamma, \quad (z_i z_{a_i})(z_j z_{b_j}) = (z_k z_{c_k}), \qquad (c_k = b_{a_k} : k = 1, 2, \ldots, n), \quad (13)$$

and similarly a *composition of orders*

$$ab = c, \quad (z_{a_1} \ldots z_{a_n})(z_{b_1} \ldots z_{b_n}) = (z_{c_1} \ldots z_{c_n}) \quad (c_k = b_{a_k} : k = 1, 2, \ldots, n). \quad (14)$$

The order $c = ab$ is derived from the order b by the same *position-permutation* as the order $a = ao$ from the order o.

Hence, just as $R^{(a)} = F^{(a)} R = F^{(a)} R^{(o)}$ (8''), so $R^{(c)} = R^{(ab)} = F^{(a)} R^{(b)}$. But $R^{(b)} = F^{(b)} R^{(o)}$ and $R^{(c)} = F^{(c)} R^{(o)}$. Thus, corresponding to (13) and (14), we have the *composition of cross-ratio transformations* $F^{(a)}$:

$$F^{(a)}F^{(b)} = F^{(ab)}, \tag{15}$$
$$R'' = F^{(a)}R', \quad R' = F^{(b)}R; \quad R'' = F^{(a)}(F^{(b)}R) = F^{(ab)}R.$$

The $n!$ substitutions a form the symmetric substitution-group $G^n_{n!}$ of order $n!$.

The $n!$ transformations $R' = F^{(a)}R$ (8″) of the flat space R_{n-3} are for $n \geqq 5$* distinct,† and form a holoedrically isomorphic transformation-group $G_{n!}$ which, from its source, I call the *cross-ratio transformation-group* $G_{n!}$ *of the flat space of* $n-3$ *dimensions* R_{n-3}.

The $(n-1)!$ collineations $F^{(a)}$ $(a_n = 1)$ form by themselves a collineation group $G_{(n-1)!}$ holoedrically isomorphic with the symmetric substitution-group $G^{n-1}_{(n-1)!}$ on the $n-1$ letters $z_1 \ldots . z_{n-1}$.

The remaining $n! - (n-1)!$ transformations $F^{(a)}$ are Cremona transformations of order $n-3$.

§2.

The collineation-group $G_{(n-1)!}$ *of the transformations* $F^{(a)}$ $(a_n = n)$ *is the* (*Klein's*) *group of* $(n-1)$ *collineations permuting amongst themselves in all possible ways certain* $n-1$ *points* $P_1, \ldots, P_{n-1}$.

We introduce the homogeneous point-coordinates x,

$$R = (r_1, \ldots, r_{n-3}) = (x_1 : x_2 : \ldots : x_{n-3} : x_{n-2}), \tag{16}$$

---

*For $n = 4$, the group is made up of the six linear fractional transformations $r' = \lambda_i(r)$ $(i = 0, 1, \ldots, 5)$. This well-known group plays a fundamental rôle in the theory of the binary quartic.

The general cross-ratio group $G_{n!}$ $(n \geqq 5)$ I obtained in 1895.

Mr. Slaught, in his forthcoming Chicago dissertation, discusses in detail the group for $n = 5$.

The group for $n = 5$ has been given (from a standpoint quite different) by Mr. S. Kantor, "Theorie der endlichen Gruppen von eindeutiger Transformationen in der Ebene" (Berlin, 1895, pp. 11, 19, 51, 52, 105). And Mr. Kantor may have somewhere indicated the generalization to $n = n$.

†If the two transformations $F^{(a)}$, $F^{(b)}$ are identical: $F^{(a)}R = F^{(b)}R$ for every point R, then denoting by a' the order reciprocal to a $(a'a = o)$ and by c the product $a'b$ one has $F^{(c)}R = F^{(o)}R$, that is, when one lets the point R depend as in (9, 3) upon the n variables z_i (1),

$$[z_{c_n} z_{c_{n-1}} z_{c_{n-2}} z_{c_i}] = [z_n z_{n-1} z_{n-2} z_i], \qquad (i = 1, \ldots, n-3).$$

The z's being general variables, this implies for every i that the two tetrads $z_{c_n} z_{c_{n-1}} z_{c_{n-2}} z_{c_i}$, $z_n z_{n-1} z_{n-2} z_i$ are the same order apart, whence the fixed triads $z_{c_n} z_{c_{n-1}} z_{c_{n-2}}$, $z_n z_{n-1} z_{n-2}$ are the same and so the residual z_{c_i}, z_i. Thus $c_i = i$ for $i = 1, \ldots, n-3$, and, indeed (by e), for $i = 1, \ldots, n$. That is, the orders c and o are the same, and likewise the orders a and b.

where
$$r_i = x_i/x_{n-2}, \qquad (i = 1, 2, \ldots, n-3). \quad (17)$$

It is convenient to introduce also the symbols x_{n-1}, x_n with the respective values $x_{n-1} = 0$, $x_n = \infty$. Then, remembering that $r_{n-2} = 1$, $r_{n-1} = 0$, $r_n = \infty$, we have
$$r_i = x_i/x_{n-2}, \qquad (i = 1, \ldots, n). \quad (17')$$

We introduce also the supernumerary homogeneous point-coordinates y:
$$y_i = \sum_{g=1}^{g=n-1} x_g - (n-1)x_i, \quad (n-1)x_i = y_{n-1} - y^i \quad (18)$$
$$(i = 1, \ldots, n-1)$$

with the identity
$$\sum_{i=1}^{i=n-1} y_i \equiv 0. \quad (19)$$

The collineation $F^{(a)}(a_n = n)$ (11_1) is then
$$\mu x_i' = x_{a_i} - x_{a_{n-1}}, \qquad (i = 1, 2, \ldots, n-2), \quad (20)$$

or
$$\mu y_i' = y_{a_i}, \qquad (i = 1, 2, \ldots, n-1), \quad (20')$$

where μ is a proportionality-factor.

The group $G_{(n-1)!}$ of $(n-1)!$ collineations $F^{(a)}(a_n = n)$ is then Klein's group.* It permutes amongst themselves in all possible ways the $n-1$ points $P_j\,(j = 1, 2, \ldots, n-1)$:

$$(y_1 : \ldots : y_i : \ldots : y_j : \ldots : y_{n-1}) = (1 : \ldots : 1 : \ldots : -(n-2) : \ldots : 1), \quad (21)$$
$$(x_1 : \ldots : x_i : \ldots : x_j : \ldots : x_{n-2}) = (0 : \ldots : 0 : \ldots : \quad 1 \quad : \ldots : 0), \quad (j \neq n-1),$$
$$(x_1 : \ldots : x_i : \ldots\ldots\ldots : x_{n-2}) = (1 : \ldots : 1 : \ldots\ldots\ldots\ldots\ldots : 1), \quad (j = n-1),$$

which are more conveniently given by the use of Kronecker's symbol δ_{st} with the definition:
$$\delta_{st} = 0 \ (s \neq t), \quad \delta_{st} = \delta_{ss} = 1 \ (s = t) \quad (22)$$

---

* Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346–358, 1871).

By a change of coordinate-system, this may be written

$$\mu x_i = \sigma_1^{n-2-i}\sigma_2^{i-1}, \qquad (i = 1,\ 2, \ldots, n-2). \qquad (29)$$

This normal form of the equation of the curve shows that just as to every parameter-value σ one point of the curve corresponds, so also to every point of the curve one parameter-value corresponds.

The flat R_{n-4}

$$\sum_{i=1}^{i=n-2} l_i x_i = 0 \qquad (30)$$

meets the curve in the $n-3$ points corresponding to the $n-3$ roots $\sigma = \sigma_1/\sigma_2$ of the equation

$$\sum_{i=1}^{i=n-2} l_i \sigma_1^{n-2-i}\sigma_2^{i-1} = 0, \qquad (31)$$

and, conversely, any $n-3$ points of the curve lie in one and only one flat R_{n-4}.

This normal coordinate-system is fully defined by the curve and the three points $\sigma = \infty$, 0, 1 on the curve; the $(n-4)$-flat $x_i = 0$ is *the* flat meeting the curve in the point $\sigma = 0$ $n-2-i$ times, and in the point $\sigma = \infty$ $i-1$ times; the unit-point $x_1 = \ldots = x_{n-2}$ is the point $\sigma = 1$.

By a linear homogeneous substitution on the σ_1, σ_2, i. e., by a linear fractional substitution on the parameter σ, we change the distribution of the parameter-values over the points of the curve. There are ∞^3 such distributions and correspondingly ∞^3 normal coordinate-systems.

If a curve is given twice, once with the parameter σ and once with the parameter τ, then

$$\sigma = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \qquad (\alpha\delta - \beta\gamma \neq 0), \qquad (32)$$

i. e., the ∞^3 distributions of the preceding remark exhaust all possible distributions. For, at least, we can so determine $\alpha\beta\gamma\delta$ that τ', where

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta},$$

shall be ∞, 0, 1 at the same points at which σ is ∞, 0, 1. Then the two distributions σ, τ' lead to the same normal coordinate-system, and hence, indeed, $\sigma = \tau'$.

Hence for *every* parameter-distribution the cross-ratio of the parameters of any certain four points of the curve is the same, and we speak of the *cross-ratio of four points of the curve.*

By a collineation of the R_{n-3}, which throws one normal coordinate-system into another, the curve is thrown into itself; on the curve, the point $\sigma = \sigma_0$ (with respect to a *fixed* parameter-distribution) is thrown to the point $\sigma = \sigma_0'$, where $\sigma_0' = (\alpha\sigma_0 + \beta)/(\gamma\sigma_0 + \delta)$. These ∞^3 collineations of the curve into itself are the only[*] ones.

Through any system of n linearly independent points Q_j $(j = 1, 2, \ldots, n)$ of the R_{n-3} one and only one rational curve $S_{1,\,n-3,\,n-3}$ passes. By proper determination of the coordinate-system, we may set:

$$\begin{aligned} Q_j:&\quad x_1 : \ldots : x_i : \ldots : x_j : \ldots : x_{n-2} = 0 : \ldots : 0 : \ldots : 1 : \ldots : 0, \quad (j = 1, 2, \ldots, n-2), \\ Q_{n-1}:&\quad x_1 : \ldots : x_i : \ldots\ldots : x_{n-2} = 1 : \ldots : 1 : \ldots\ldots : 1, \\ Q_n:&\quad x_1 : \ldots : x_i : \ldots\ldots : x_{n-2} = \xi_1 : \ldots : \xi_i : \ldots\ldots : \xi_{n-2}, \end{aligned} \tag{33}$$

where no ξ is 0 and no two ξ's are equal. Let the parameter σ on the rational curve take the value $\sigma = \lambda_j$ at Q_j $(j = 1, 2, \ldots, n)$. We fix the parameter σ by the conditions $\lambda_n = \infty$, $\lambda_{n-1} = 0$, $\lambda_{n-2} = 1$. Setting

$$G(\sigma_1, \sigma_2) = \prod_{j=1}^{j=n-2} (\sigma_1 - \lambda_j \sigma_2), \quad G_i(\sigma_1, \sigma_2) = g_i\, G(\sigma_1, \sigma_2)/(\sigma_1 - \lambda_i \sigma_2), \quad (i = 1, 2, \ldots, n-2), \tag{34}$$

where the g_i's are constants, we have as parametric representation of the most general rational curve $S_{1,\,n-3,\,n-3}$ containing the points Q_j with the respective parametric values $\sigma = \lambda_j$ $(j = 1, 2, \ldots, n-2)$,

$$\mu x_i = G_i(\sigma_1, \sigma_2), \qquad (i = 1, 2, \ldots, n-2). \tag{35}$$

The condition that the curve-point $\sigma = \lambda_{n-1} = 0$ shall be Q_{n-1} is $\mu' g_i = \lambda_i$ $(i = 1, 2, \ldots, n-2)$, and the condition that the curve-point $\sigma = \lambda_n = \infty$ shall be Q_n is $\mu'' g_i = \xi_i$ $(i = 1, 2, \ldots, n-2)$. We set then in (34, 35)

$$g_i = \xi_i, \quad \lambda_i = \xi_i/\xi_{n-2}, \qquad (i = 1, 2, \ldots, n-2), \tag{36}$$

and have, indeed, in (35) *the single rational curve* $S_{1,\,n-3,\,n-3}$ which contains the n points Q_j $(j = 1, 2, \ldots, n)$. (That the determinant $|g_{ij}| \neq 0$ is a consequence of the linear independence of the points Q_j.)

---

[*] This theorem is given, for instance, by Meyer, "Apolarität und rationale Curven," p. 398, 1883.

where
$$r_i = x_i / x_{n-2}, \qquad (i = 1, 2, \ldots, n-3). \tag{17}$$

It is convenient to introduce also the symbols x_{n-1}, x_n with the respective values $x_{n-1} = 0$, $x_n = \infty$. Then, remembering that $r_{n-2} = 1$, $r_{n-1} = 0$, $r_n = \infty$, we have
$$r_i = x_i / x_{n-2}, \qquad (i = 1, \ldots, n). \tag{17'}$$

We introduce also the supernumerary homogeneous point-coordinates y:
$$y_i = \sum_{g=1}^{g=n-1} x_g - (n-1)\, x_i, \quad (n-1)\, x_i = y_{n-1} - y^i \qquad (i = 1, \ldots, n-1) \tag{18}$$
with the identity
$$\sum_{i=1}^{i=n-1} y_i \equiv 0. \tag{19}$$

The collineation $F^{(a)}(a_n = n)$ (11_1) is then
$$\mu x_i' = x_{a_i} - x_{a_{n-1}}, \qquad (i = 1, 2, \ldots, n-2), \tag{20}$$
or
$$\mu y_i' = y_{a_i}, \qquad (i = 1, 2, \ldots, n-1), \tag{20'}$$
where μ is a proportionality-factor.

The group $G_{(n-1)!}$ of $(n-1)!$ collineations $F^{(a)}(a_n = n)$ is then Klein's group.* It permutes amongst themselves in all possible ways the $n-1$ points $P_j\,(j = 1, 2, \ldots, n-1)$:
$$\begin{aligned}
(y_1 : \ldots : y_i : \ldots : y_j : \ldots : y_{n-1}) &= (1 : \ldots : 1 : \ldots : -(n-2) : \ldots : 1), && (21)\\
(x_1 : \ldots : x_i : \ldots : x_j : \ldots : x_{n-2}) &= (0 : \ldots : 0 : \ldots : \quad 1 \quad : \ldots : 0), && (j \neq n-1),\\
(x_1 : \ldots : x_i : \ldots\ldots\ldots : x_{n-2}) &= (1 : \ldots : 1 : \ldots\ldots\ldots\ldots\ldots : 1), && (j = n-1),
\end{aligned}$$
which are more conveniently given by the use of Kronecker's symbol δ_{st} with the definition:
$$\delta_{st} = 0 \ (s \neq t), \quad \delta_{st} = \delta_{ss} = 1 \ (s = t) \tag{22}$$

---

* Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346–358, 1871).

the *composition of cross-ratio transformations* $F^{(a)}$:

$$F^{(a)}F^{(b)}=F^{(ab)}, \tag{15}$$
$$R''=F^{(a)}R',\quad R'=F^{(b)}R;\quad R''=F^{(a)}(F^{(b)}R)=F^{(ab)}R.$$

The $n!$ substitutions α form the symmetric substitution-group $G^{n}_{n!}$ of order $n!$.

The $n!$ transformations $R'=F^{(a)}R\,(8'')$ of the flat space R_{n-3} are for $n\geqq 5$* distinct,† and form a holoedrically isomorphic transformation-group $G_{n!}$ which, from its source, I call the *cross-ratio transformation-group* $G_{n!}$ *of the flat space of* $n-3$ *dimensions* R_{n-3}.

The $(n-1)!$ collineations $F^{(a)}\,(a_n=1)$ form by themselves a collineation group $G_{(n-1)!}$ holoedrically isomorphic with the symmetric substitution-group $G^{n-1}_{(n-1)!}$ on the $n-1$ letters $z_1 \ldots . z_{n-1}$.

The remaining $n!-(n-1)!$ transformations $F^{(a)}$ are Cremona transformations of order $n-3$.

§2.

The collineation-group $G_{(n-1)!}$ *of the transformations* $F^{(a)}\,(a_n=n)$ *is the (Klein's) group of* $(n-1)$ *collineations permuting amongst themselves in all possible ways certain* $n-1$ *points* $P_1, \ldots . , P_{n-1}$.

We introduce the homogeneous point-coordinates x,

$$R=(r_1, \ldots . , r_{n-3})=(x_1:x_2: \ldots . :x_{n-3}:x_{n-2}), \tag{16}$$

---

*For $n=4$, the group is made up of the six linear fractional transformations $r'=\lambda_i(r)$ $(i=0, 1, \ldots . , 5)$. This well-known group plays a fundamental rôle in the theory of the binary quartic.

The general cross-ratio group $G_{n!}\,(n\geq 5)$ I obtained in 1895.

Mr. Slaught, in his forthcoming Chicago dissertation, discusses in detail the group for $n=5$.

The group for $n=5$ has been given (from a standpoint quite different) by Mr. S. Kantor, "Theorie der endlichen Gruppen von eindeutiger Transformationen in der Ebene" (Berlin, 1895, pp. 11, 19, 51, 52, 105). And Mr. Kantor may have somewhere indicated the generalization to $n=n$.

†If the two transformations $F^{(a)}$, $F^{(b)}$ are identical: $F^{(a)}R=F^{(b)}R$ for every point R, then denoting by a' the order reciprocal to $a\,(a'a=o)$ and by c the product $a'b$ one has $F^{(o)}R=F^{(c)}R$, that is, when one lets the point R depend as in (9, 8) upon the n variables z_i (1),

$$[z_{c_n}z_{c_{n-1}}z_{c_{n-2}}z_{c_i}]=[z_n z_{n-1} z_{n-2} z_i], \qquad (i=1, \ldots . , n-3).$$

The z's being general variables, this implies for every i that the two tetrads $z_{c_n} z_{c_{n-1}} z_{c_{n-2}} z_{c_i}$, $z_n\, z_{n-1} z_{n-2}\, z_i$ are the same order apart, whence the fixed triads $z_{c_n}\, z_{c_{n-1}} z_{c_{n-2}}$, $z_n z_{n-1}\, z_{n-2}$ are the same and so the residual z_{c_i}, z_i. Thus $c_i=i$ for $i=1, \ldots . , n-3$, and, indeed (by e), for $i=1, \ldots . , n$. That is, the orders c and o are the same, and likewise the orders a and b.

§5.

The cross-ratio transformation group $G_{n!}\{F^{(a)}\}$ is thus projectively determined by its fundamental $(n-1)$-gon $P_j\,(j=1, 2, \ldots, n-1)$:

> *If P_n is any point linearly independent of the $P_j\,(j=1, 2, \ldots, n-1)$, then the transformation $F^{(a)}$, $R'=F^{(a)}R$, throws the point $R=P_n$ to the point $R'=F^{(a)}P_n$ such that the two n-gons*
>
> $$P_1P_2\ldots P_{n-1}F^{(a)}P_n,\quad P_{a_1}P_{a_2}\ldots P_{a_{n-1}}P_{a_n}$$
>
> *are projective.*

We take any system of n linearly independent points $P_i\,(i=1, 2, \ldots, n)$ of the flat R_{n-3}. Through them one and only one rational curve $S_{1,\,n-3,\,n-3}$ passes (§4). Identifying them taken in any order $a=(P_{a_1}\ldots P_{a_n})$ with the n points $(Q_1\ldots Q_n)$ of §4, $P_{a_i}=Q_i=P_i^{(a)}$, we determine the homogeneous coordinate system $(x_1:\ldots\ldots:x_{n-2})$ of §4, or say $(x_1^{(a)}:\ldots:x_{n-2}^{(a)})$, and the corresponding non-homogeneous system $(r_1^{(a)},\ldots,r_{n-3}^{(a)})$ where $r_i^{(a)}=x_i^{(a)}/x_{n-2}^{(a)}\,(i=1, 2, \ldots, n-2)$. Then P_{a_n} is say

$$P_{a_n}:\quad (x_1^{(a)}:\ldots:x_{n-2}^{(a)}=\xi_1^{(a)}:\ldots:\xi_{n-2}^{(a)}),\quad (r_1^{(a)},\ldots,r_{n-3}^{(a)}=\rho_1^{(a)},\ldots,\rho_{n-3}^{(a)}). \tag{37}$$

The parameter $\sigma^{(a)}$ of the curve shall take at P_{a_i} the value $\sigma^{(a)}=\lambda_i^{(a)}$ and in particular at P_{a_n}, $P_{a_{n-1}}$, $P_{a_{n-2}}$ the respective values $\lambda_n^{(a)}=\infty$, $\lambda_{n-1}^{(a)}=0$, $\lambda_{n-2}^{(a)}=1$. Then since by §4 (36), $\lambda_i^{(a)}=\xi_i^{(a)}/\xi_{n-2}^{(a)}$ $(i=1, 2, \ldots, n-2)$, we have

$$\lambda_i^{(a)}=\rho_i^{(a)},\qquad (i=1, 2, \ldots, n-2). \tag{38}$$

Further by (4),

$$\lambda_i^{(a)}=[\infty 0 1 \lambda_i^{(a)}]=[\lambda_n^{(a)}\lambda_{n-1}^{(a)}\lambda_{n-2}^{(a)}\lambda_i^{(a)}]=[P_{a_n}P_{a_{n-1}}P_{a_{n-2}}P_{a_i}]_{\text{(on curve)}}. \tag{39}$$

We now refer everything to the original order $o=(P_1\ldots P_n)$, and drop the (o) from the x_i, r_i, ξ_i, ρ_i, σ_i, λ_i. We further identify the $n-1$ points $P_1\ldots P_{n-1}$ with the $n-1$ fundamental points $P_1\ldots P_{n-1}$ of the cross-ratio transformation group $G_{n!}$ (§2).

The $x_i^{(a)}$'s are expressed in the x_j's identically thus:

$$\mu\, x_i^{(a)}\equiv\sum_{j=1}^{j=n-2} q_{ij}^{(a)}\, x_j,\qquad (i=1, \ldots, n-2), \tag{40}$$

where μ is a proportionality factor and the $q_{ij}^{(a)}$'s are constants whose determinant is not 0.

The collineation

$$\mu x_i' = \mu x_i^{(a)} = \sum_{j=1}^{j=n-2} q_{ij}^{(a)} x_j, \quad r_i' = r_i^{(a)}, \qquad (i = 1, 2, \ldots, n-2) \tag{41}$$

throws P_{a_i} *to* $P_i\,(i = 1, 2, \ldots, n-1)$, *and*, as I shall prove, P_{a_n} *to* $F^{(a)}P_n$.

In fact, since (§4) the cross-ratio of four points on the curve is independent of the choice of parameter, we have

$$\lambda_i^{(a)} = [P_{a_n} P_{a_{n-1}} P_{a_{n-2}} P_{a_i}]_{(\text{on curve})} = [\lambda_{a_n} \lambda_{a_{n-1}} \lambda_{a_{n-2}} \lambda_{a_i}] = f_i^{(a)}(\lambda_1, \lambda_2, \ldots, \lambda_{n-3}), \qquad (i = 1, 2, \ldots, n-3). \tag{42}$$

Further, since $\lambda_i^{(a)} = \rho_i^{(a)}$ and $\lambda_i = \rho_i$, the collineation (41):

$$r_i' = r_i^{(a)}, \qquad (i = 1, 2, \ldots, n-3), \tag{41}$$

does indeed throw the point

$$R = P_{a_n}: \qquad r_i^{(a)} = \rho_i^{(a)} \qquad (i = 1, 2, \ldots, n-3) \tag{43}$$

to the point

$$R' = F^{(a)}P_n: \qquad r_i' = f_i^{(a)}(\rho_1, \rho_2, \ldots, \rho_{n-3}), \qquad (i = 1, 2, \ldots, n-3). \tag{44}$$

§6.

The fixed points of the transformations $F^{(a)}$ and binary n-ic forms with collineations into themselves.

We consider (only) the fixed points which are linearly independent of the $n-1$ fundamental points $P_1 \ldots P_{n-1}$ of the group $G_{n!}$,

If $R = P_n$ is such a fixed point for the transformation $F^{(a)}$, then (§5) we have the projectivity

$$[P_1 P_2 \ldots P_{n-1} P_n] \barwedge [P_{a_1} P_{a_2} \ldots P_{a_{n-1}} P_{a_n}]. \tag{45}$$

The collineation (41) throws the n-gon into itself, and hence the rational curve $S_{1,\,n-2,\,n-3}$ through the n points P_i into itself, and on the curve is equivalent to a collineation of the fundamental parameter σ. To the n points P_i corresponds then a binary n-ic form in $\sigma_1, \sigma_2\,(\sigma = \sigma_1/\sigma_2)$ with a collineation into itself.

Conversely, to every binary n-ic form $H(\sigma_1, \sigma_2)$ $(n > 4)$ of non-vanishing discriminant invariant (up to a constant factor) under a group of d binary collineations G_d', where then d is a factor of $n!$, corresponds $n!/d$ points P_n in the R_{n-3} each invariant under a subgroup G_d of d transformations $F^{(a)}$ of the cross-ratio group $G_{n!}$; these $n!/d$ points and subgroups are conjugate under the main group. We determine these points P_n by linearly transforming $H(\sigma_1, \sigma_2)$ in all possible ways into a *normal form* $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2})$ with the factor $\overline{\sigma_1}, \overline{\sigma_2}(\overline{\sigma_1}-\overline{\sigma_2})$. If $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2}) \equiv \overline{\sigma_1}\,\overline{\sigma_2}(\overline{\sigma_1}-\overline{\sigma_2}) \prod_{i=1}^{i=n-3} (\overline{\sigma_1}-\lambda_i\overline{\sigma_2})$, then the point $\mu_i=\lambda_i\,(i=1, 2, \ldots, n-3)$ is such a point P_n.

Obviously if the λ_i's of the normal form $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2})$ depend upon certain t arbitrary parameters $\theta_1, \ldots, \theta_t\,(t<n-3)$, then the locus of the point $P_n(\theta_1, \ldots, \theta_t)$ is a t-fold spread fixed by points under the corresponding group G_d of d transformations $F^{(\alpha)}$.

Now Klein (e. g., in his "Vorlesungen über das Ikosaeder . . . ," 1884) has determined all finite groups of binary collineations and the corresponding systems of invariant binary forms. In order to apply the preceding theory for any particular value of n, it is then comparatively easy to determine* the binary n-ic forms of non-vanishing discriminant with collineations into themselves.

If we consider only the collineations $F^{(\alpha)}\,(\alpha_n=n)$, we have in this §6 results connecting the fixed points of Klein's group of $(n-1)!$ collineations in R_{n-3} with the binary n-ic forms invariant under binary collineations for each of which one of the fixed points is a zero point of the n-ic form.

THE UNIVERSITY OF CHICAGO.

---

*See, for instance, Bolza, "On Binary Sextics with Linear Transformations into Themselves" (American Journal of Mathematics, vol. 10, pp. 47–70).

For $n=5$ the results may be compared with those obtained directly by Mr. Slaught.

Asymptotic Evaluation of certain Totient Sums.

By Derrick Norman Lehmer.

Introduction.

This investigation is the outcome of an attempt to account for what seems to be a remarkable law first observed in particular cases in 1895. It may be stated as follows:

Consider any set s of k linear forms, $an+b_i$ $(i=1\ \ldots .\ k)$, all of which have the same modulus a, and where $[a, b_i]=1$.* Consider, further, a function $\Theta_s(x)$ such that $\Theta_s(x)=1$ or 0, according as each of the prime divisors of x belongs to one of the forms of the set s or not. If then $\nu(x)$ denotes the number of distinct primes in x, we have

$$\lim_{N=\infty}\frac{\sum\limits_{x=1}^{N}2^{\nu(x)}\Theta_s(x)}{N}=\text{constant}.$$

In the following we shall prove this law where s is the set of linear forms belonging to a binary quadratic form. We shall also determine the constant in this case.

In the investigation of this law, it seemed necessary to construct a more general theory of what Professor Sylvester has called the Totient Function—the function which denotes the number of integers not greater than a given number and prime to it. Euler, the first to discuss the function, denotes it by $\pi(x)$. Sylvester denotes it by $\tau(x)$. Most continental writers follow Gauss, and denote it by $\phi(x)$. We shall denote it by $\phi_1(x)$, as a special case of a more general function $\phi_m(x)$, which we have called the Multiple Totient of x of multiplicity m, or the m-fold totient of x.

---

* $[a, b]$ here, as always, denotes the greatest common divisor of a and b.

CHAPTER I.

MULTIPLE TOTIENTS.

The ordinary totient of an integer x being defined as the number of integers y such that

$$x \geqq y \geqq 1$$

and
$$[x, y] = 1,$$

we have as a generalization the following definition:

DEFINITION: *The m-fold totient of an integer x is the number of different sets of integers*

$$x_1 x_2 \ldots . x_m$$

which satisfy the conditions

$$x \geqq x_i \geqq 1, \qquad (i = 1 \ldots . m)$$

and
$$[x, x_1, x_2, \ldots . , x_m] = 1.$$

Two sets are considered different unless they contain the same integers arranged in th same order.

Let x be given in terms of its component primes

$$x = \prod_{i=1}^{r} p_i^{a}.$$

Disregard for the moment the second condition. The number of sets is then x^m since each of the m elements x_i may run independently through the value 1, 2, ... x. Now, among these values there are $\frac{x}{p_i}$ multiples of p_i. Ther will then be $\left(\frac{x}{p_i}\right)^m$ sets where each element is a multiple of p_i. Similarly, ther will be $\left(\frac{x}{\prod_{i=1}^{s} p_i}\right)^m$ sets where each element is a multiple of $\prod_{i=1}^{s} p_i$. The familia principle of cross-classification* gives as the required number of sets $x^m \prod_{i=1}^{r}\left(1-\frac{1}{p_i^m}\right)$, which is the formula for the m-fold totient of x, for $x \neq 1$

---

* See note on the principle of cross-classification at the end of this chapter.

For $x = 1$ we have the m-fold totient equal to unity. Denoting it by $\phi_m(x)$, we have the theorem:

THEOREM I. *For* $x = \prod_{i=1}^{r} p_i^{\alpha_i} \neq 1$,

$$\phi_m(x) = x^m \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^m}\right),$$

and

$$\phi_m(1) = 1.$$

The formula for $\phi^m(x)$ may be written in different forms easily obtainable from this. Thus:

$$\phi_m(x) = \prod_{i=1}^{r} p_i^{m(\alpha_i - 1)}(p_i^m - 1),$$

with again the understanding that $\phi_m(1) = 1$. A third important form is given in the theorem:

THEOREM II.

$$\phi_m(x) = x^m \sum_{(d)} \frac{\mu(d)}{d^m},$$

the sum extending over all d's which are divisors of x, *and* μ *being Mertens's Function (cf. Crelle's Journal, vol.* LXXVII, p. 289, 1874), *defined as follows:*

$$\mu(1) = 1,$$

and

$$\mu(x) = (-1)^\lambda \text{ or } 0,$$

according as each of the λ *distinct primes in* x *occurs to the first power or not.*

We shall see that for every theorem connected with the ordinary totient $\phi_1(x)$ there is a corresponding one in the theory of the function $\phi_m(x)$. The following theorem is evident:

THEOREM III. *If* x *and* y *are relative primes,*

$$\phi_m(x)\,\phi_m(y) = \phi_m(xy).$$

Also we can show the following

THEOREM IV.

$$\sum_{(d)} \phi_m(d) = x^m,$$

the sum extending over all d's which are divisors of x.

For we have, when $x > 1$,

$$\phi_m(x) = \prod_{i=1}^{r} (p_i^{m\alpha_i} - p_i^{m(\alpha_i - 1)});$$

and the sum in question may be written

$$\prod_{i=1}^{r} \sum_{k=0}^{\alpha_i} \phi_m(p_i^k).$$

Now for $k = 0$, $\phi_m(p_i^k) = 1$. Otherwise, it is given by the formula above as

$$p_i^{mk} - p_i^{m(k-1)}.$$

The terms of the sum are then seen to cancel in pairs except the last one, which is $p_i^{\alpha_i m}$. The expression is then equal to $\prod_{i=1}^{r} p_i^{\alpha_i m} = x^m$. Hence the theorem.

The following theorem is of importance also:

THEOREM V. $\phi_m(x^n) = x^{m(n-1)} \phi_m(x)$,

We have
$$\phi_m(x^n) = x^{mn} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^m}\right)$$

and
$$\phi_m(x) = x^m \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^m}\right).$$

By division the theorem follows.

We shall need to express $\phi_m(xy)$ in terms of $\phi_m(y)$. To this end we prove the fundamental theorem:

THEOREM VI.

$$\phi_m(xy) = \prod_{i=1}^{r} [p_i^{m\alpha_i} - p_i^{m(\alpha_i - 1)} \lambda(y, p_i)] \phi_m(y),$$

where $\lambda(y, p_i) = 0$ *or* 1, *according as* p_i *is or is not a divisor of* y.

Let $y = l \prod_{i=1}^{r} p_i^{\beta_i}$, where $[l, x] = 1$, and $\beta_i \geqq 0$.

We have
$$\phi_m(xy) = \prod_{i=1}^{r} [p_i^{m(\alpha_i + \beta_i - 1)} (p_i^m - 1)] \phi_m(l);$$

$$= \prod_{i=1}^{r} [p_i^{m(\beta - 1)} (p_i^m - 1)] \prod_{i=1}^{r} p_i^{m\alpha_i} \phi_m(l).$$

If, now, $\beta_i \neq 0$, the expression $p_i^{m(\beta_i-1)}(p_i^m-1)\phi_m(l)$ is equal to $\phi_m(p_i^{\beta_i} l)$. If, however, $\beta_i = 0$, this change cannot be made, but in this event p_i is not a factor of y. Thus the product of the $p_i^{\beta_i}$'s that do go over into the ϕ_m function is, when multiplied by l, precisely y itself. The factor left outside when $\beta_i = 0$ is $p_i^{m(\alpha_i-1)}(p_i^m-1)$. If $\beta_i \neq 0$, the factor is $p_i^{m\alpha_i}$, and the theorem is proved.

It follows from this theorem that if y runs through the values 1, 2, 3, ... , the coefficient of $\phi_m(y)$ on the right will be periodic of period $\prod_{i=1}^{r} p_i$. For if $y \equiv y' \left(\bmod \prod_{i=1}^{r} p_i\right)$, then p_i divides y when, and only when it divides y' or

$$\lambda(y, p_i) = \lambda(y', p_i),$$

For example, in the case of the ordinary totient, $(m = 1)$, we have for $x = 18$:

$$\begin{aligned}
\phi_1(18.1) &= 6\,\phi_1(1),\\
\phi_1(18.2) &= 12\,\phi_1(2),\\
\phi_1(18.3) &= 9\,\phi_1(3),\\
\phi_1(18.4) &= 12\,\phi_1(4),\\
\phi_1(18.5) &= 6\,\phi_1(5),\\
\phi_1(18.6) &= 18\,\phi_1(6).
\end{aligned}$$

The coefficients now recur, the period being $2.3 = 6$.

We define now two functions by the following equations:

$$\Phi_m(x, n, k) = \sum_{i=1}^{\left[\frac{x}{k}\right]} \phi_m(i^n k^n),$$

and

$$\Omega_m(x, n, k) = \sum_{i=1}^{\left[\frac{x}{k}\right]} \frac{\phi_m(ik)}{(i^n k^n)^m},$$

where m, n and k are positive integers greater than zero, and x is positive but not necessarily integral. $\left[\frac{x}{k}\right]$ denotes here, as always, the greatest integer in $\frac{x}{k}$. In studying these functions we shall also need the function $S(x, k)$ defined by the equation

$$S(x, k) = \sum_{i=1}^{[x]} i^k.$$

We shall also need the well-known theorem,

$$\left[\frac{[x]}{k}\right] = \left[\frac{x}{k}\right].$$

THEOREM VII.

$$\sum_{j=1}^{[x]} j^{m(n-1)} \Phi_m\left(\frac{x}{j}, n, 1\right) = S(x, mn).$$

To show this theorem, we take the first difference of the function on the left with respect to $[x]$. This may be written

$$\sum_{j=1}^{[x]} j^{m(n-1)} \left\{ \Phi_m\left(\frac{x}{j}, n, 1\right) - \Phi_m\left(\frac{x-1}{j}, n, 1\right) \right\}.$$

Now, if j does not divide $[x]$, the expression in the braces will vanish. If, on the other hand, j is a divisor of $[x]$, d say,—then the expression in braces becomes

$$\phi_m\left(\frac{[x]^n}{d^n}\right).$$

But this, by Theorem V, is equal to

$$\frac{[x]^{m(n-1)}}{d^{m(n-1)}} \phi_m\left(\frac{[x]}{d}\right).$$

The first difference in question may then be written:

$$\sum_{(d)} [x]^{m(n-1)} \phi_m\left(\frac{[x]}{d}\right),$$

where the sum is extended over all d's which are divisors of $[x]$. If we put $dd' = [x]$, this becomes

$$\sum_{(d')} [x]^{m(n-1)} \phi_m(d'),$$

where again the sum extends over all the divisors, d', of $[x]$. By Theorem IV this is $[x]^{mn}$. Since now $\Phi_m(1, n, 1) = 1^{mn}$, the theorem follows.

In precisely the same way we can prove

THEOREM VIII.

$$\sum_{j=1}^{[x]} \frac{1}{j^{mn}} \Omega_m\left(\frac{x}{j}, n, 1\right) = S(x, -m(n-1)).$$

Making use of the well-known recursion formula for Mertens's Function (cf. Bachmann, "Zahlentheorie," vol. II, p. 310), we easily derive from the formula of Theorem VII the following:

THEOREM IX.

$$\Phi_m(x, n, 1) = \sum_{i=1}^{[x]} \mu(i)\, i^{m(n-1)} S\left(\frac{x}{i}, mn\right).$$

This theorem is of cardinal importance for the sequel, as is also the corresponding theorem for the Ω function, which is proved in precisely the same way starting from Theorem VIII.

THEOREM X.

$$\Omega_m(x, n, 1) = \sum_{i=1}^{[x]} \frac{\mu(i)}{i^{mn}} S\left(\frac{x}{i}, -m(n-1)\right).$$

Similar formulæ for the general case where $k > 1$ do not appear to be obtainable, but we may find a general reduction formula by which we may reduce the general formulæ.

THEOREM XI. If $k = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $k' = \frac{k}{p_r^{\alpha_r}}$, *then*

$$\Phi_m(x, n, k) = p_r^{m(\alpha_r n - 1)} (p_r - 1)\, \Phi_m\left(\frac{x}{p_r^{\alpha_r}}, n, k'\right) + p_r^{m(\alpha_r n - 1)}\, \Phi_m\left(\frac{x}{p_r^{\alpha_r}}, n, p_r k'\right).$$

To prove this formula, we observe that when i is prime to p_r, we have, by Theorem VI,

$$\phi_m(i^n k^n) = p_r^{m(\alpha_r n - 1)} (p_r - 1)\, \phi_m(i^n k'^n).$$

Assuming for the moment that i is prime to p_r for $i = 1, 2, \ldots \left[\frac{x}{k}\right]$, we get on the left by summing

$$\sum_{i=1}^{\left[\frac{x}{k}\right]} \phi_m(i^n k^n) = \Phi_m(x, n, k),$$

by definition. On the right, we have

$$p_r^{m(\alpha_r n - 1)} (p_r - 1) \sum_{i=1}^{\left[\frac{x}{k}\right]} \phi_m(i^n k'^n),$$

Now, since

$$\left[\frac{x}{k}\right] = \left[\frac{\left[\frac{x}{p_r^{a_r}}\right]}{k'}\right],$$

this last is by definition

$$p_r^{m(a_r n-1)}(p_r - 1)\,\Phi_m\left(\frac{x}{p_r^{a_r}},\, n,\, k'\right).$$

We must now correct for those terms where i is not prime to p_r; that is, for

$$i = p_r,\ 2p_r,\ 3p_r,\ \ldots\ldots,\ \left[\frac{\left[\frac{x}{k}\right]}{p_r}\right]p_r.$$

For such terms we have, by Theorem VI.

$$\phi_m(i^n k^n) = p_r^{m a_r n}\,\phi_m(i^n k'^n).$$

But we have already taken such terms with a coefficient $p_r^{m(a_r n-1)}(p_r - 1)$, so we have only to add the sum

$$p_r^{m(a_r n-1)} \sum_{(i)} \phi_m(i^n k'^n);$$

where the sum is extended over the values of i as given above, viz.:

$$i = p_r,\ 2p_r\ \ldots\ldots\left[\frac{\left[\frac{x}{k}\right]}{p_r}\right]p_r.$$

We might write this sum,

$$p_r^{m(a_r n-1)} \sum_{(i)} \phi_m(i^n p_r^n k'^n),$$

where now i takes the values

$$i = 1,\ 2,\ 3,\ \ldots\ldots\left[\frac{\left[\frac{x}{k}\right]}{p_r}\right].$$

that is,

$$i = 1,\ 2,\ 3,\ \ldots\ldots\left[\frac{\left[\frac{x}{p_r^{a_r}}\right]}{p_r k'}\right].$$

Let, now, x be any positive number. Then we have

$$S(x,\, n) = \frac{[x]^{n+1}}{n+1} + \Delta_{S(x,\, n)},$$

as above. Put now $[x] = x - \sigma_x$, where $0 \leqq \sigma_x < 1$, and we have, by Taylor's theorem,

$$\frac{[x]^{n+1}}{n+1} = \frac{(x-\sigma_x)^{n+1}}{n+1} = \frac{x^{n+1}}{n+1} - \sigma_x (x - \theta\sigma_x)^n,$$

where $0 \leqq \theta \leqq 1$. We have, therefore,

$$S(x,\, n) = \frac{x^{n+1}}{n+1} + \Delta'_{S(x,\, n)},$$

where
$$\Delta'_{S(x,\, n)} = \Delta_{S(x,\, n)} - \sigma_x (x - \theta\sigma_x)^n.$$

Now, both terms on the right are essentially positive, and their difference is less than the greater of them. The greatest value of $\sigma_x (x - \theta\sigma_x)^n$ is obtained by putting $\sigma_x = 1$, $\theta = 0$, which gives x^n. The theorem follows.

THEOREM II. *For* $x \geqq 1$,

$$S(x,\, -1) = \log x + \Delta_{S(x,\, -1)},$$

where
$$|\Delta_{S(x,\, -1)}| \leqq 4.$$

We have a well-known formula for $S(x,\, -1)$, (cf. Boole, "Finite Differences," p. 93), when x is integral, namely:

$$S(x,\, -1) = \log x + \Delta_{S(x,\, -1)},$$

where
$$\Delta_{S(x,\, -1)} = \varepsilon + \frac{1}{2x} + \sum_{i=1}^{\infty} (-1)^i \frac{B_{2i-1}}{(2i)\,!} \, \frac{1}{(2i)\, x^{2i}}.$$

(ε denotes Euler's constant .577215 ...., and the B's are Bernaulli's numbers.)

It is seen at once that

$$|\Delta_{S(x,\, -1)}| \leqq \varepsilon + \tfrac{1}{2} + \sum_{i=1}^{\infty} \frac{B_{2i-1}}{(2i)\,!} \, \frac{1}{2i}.$$

But we also have (Boole, "Finite Differences," p. 109),

$$\frac{B_{2i-1}}{(2i)!} = \frac{2}{(2\pi)^{2i}} \sum_{j=1}^{\infty} \frac{1}{j^{2i}} \leq \frac{2}{(2\pi)^{2i}} \; \frac{\pi^2}{6} .$$

Also, since $(2\pi)^{2i} > \frac{\pi^2}{6} i$, we have

$$\frac{B_{2i-1}}{(2i)!} < \frac{2}{i} ;$$

so that

$$|\Delta_{S(x,\,-1)}| \leq \epsilon + \tfrac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{i^2} , \\ \leq 3 .$$

In case x is not an integer, we have as above,

$$S(x,\,-1) = \log\,[x] + \Delta_{S(x,\,-1)} ,$$

and putting $[x] = x - \sigma_x$, where $0 \leq \sigma_x < 1$,

$$S(x,\,-1) = \log x - \log\left(1 - \frac{\sigma_x}{x}\right) + \Delta_{S(x,\,-1)} .$$

We wish to examine the absolute value of

$$\log\left(1 - \frac{\sigma_x}{x}\right) , \text{ or of } \log\left(1 - \frac{\sigma_x}{[x] + \sigma_x}\right) .$$

Now, for $[x] \geq 1$, we have

$$\left|\log\left(1 - \frac{\sigma_x}{[x] + \sigma_x}\right)\right| \leq \left|\log\left(1 - \frac{\sigma_x}{1 + \sigma_x}\right)\right| ,$$

and since $\sigma_x < 1$,

$$\left|\log\left(1 - \frac{\sigma_x}{[x] + \sigma_x}\right)\right| \leq \log \tfrac{1}{2} < 1 ,$$

and the theorem follows.

THEOREM III. *For* $x \geq 1$, *and* $n > 1$,

$$S(x,\,-1) = D_{(n)} + \Delta_{S(x,\,-n)} ,$$

where

$$D_{(n)} = \sum_{j=1}^{\infty} \frac{1}{j^n} , \text{ and } |\Delta_{S(x,\,-n)}| \leq \frac{1}{[x]} .$$

We have

$$\Delta_{S(x,\,-n)} = -\sum_{j=[x]+1}^{\infty} \frac{1}{j^n} ,$$

so

$$|\Delta_{S(x,-n)}| \leqq \sum_{j=[x]+1}^{\infty} \frac{1}{j(j-1)} \leqq \frac{1}{[x]}.$$

We can now find an approximate formula for our function

$$\Phi_m(x, n, 1).$$

THEOREM IV.

$$\Phi_m(x, n, 1) = \frac{x^{mn+1}}{mn+1} \, \frac{1}{D_{(m+1)}} + \Delta\Phi_m(x, n, 1),$$

where

$$D_{(m+1)} = \sum_{j=1}^{\infty} \frac{1}{j^{n+1}} \text{ and } |\Delta\Phi_m(x, n, 1)| \leqq A x^{mn} \log x,$$

A being finite and independent of x, m and n.

We start from Theorem X of the preceding chapter,

$$\Phi_m(x, n, 1) = \sum_{i=1}^{[x]} \mu(i) i^{m(n-1)} S\left(\frac{x}{i}, mn\right).$$

Writing in our formula for $S\left(\frac{x}{i}, mn\right)$, this becomes equal to $M+N$, where

$$M = \sum_{i=1}^{[x]} \frac{\mu(i) x^{mn+1}}{(mn+1) i^{m+1}}$$

and

$$N = \sum_{i=1}^{[x]} \mu(i) i^{m(n-1)} \Delta_S\left(\frac{x}{i}, mn\right).$$

since now

$$\left|\Delta_S\left(\frac{x}{i}, mn\right)\right| \leqq \frac{x^{mn}}{i^{mn}},$$

we have, putting $\mu(i) = 1$,

$$|N| \leqq x^{mn} \sum_{i=1}^{[x]} \frac{1}{i^m}.$$

Since $m \geqq 1$,

$$|N| \leqq x^{mn} \sum_{i=1}^{[x]} \frac{1}{i},$$

$$\leqq A_1 x^{mn} \log x,$$

where A_1 is finite and independent of x, m and n.

We break up M into two parts:

$$M = P + Q,$$

where

$$P = \frac{x^{mn+1}}{mn+1} \sum_{i=1}^{\infty} \frac{\mu(i)}{i^{m+1}},$$

$$= \frac{x^{mn+1}}{mn+1} \, \frac{1}{D_{(m+1)}},$$

and

$$Q = \frac{-x^{mn+1}}{mn+1} \sum_{i=[x]+1}^{\infty} \frac{\mu(i)}{i^{m+1}}.$$

Now

$$|Q| \leq \frac{x^{mn+1}}{mn+1} \sum_{i=[x]+1}^{\infty} \frac{1}{i^{2}},$$

$$\leq \frac{x^{mn+1}}{mn+1} \sum_{i=[x]+1}^{\infty} \frac{1}{i(i-1)},$$

$$\leq \frac{x^{mn+1}}{mn+1} \, \frac{1}{[x]},$$

$$\leq A_2 x^{mn},$$

$$\leq A_2 x^{mn} \log x,$$

where A_2 is finite and independent of x, m and n. The theorem follows.

THEOREM V. *For any prime $p > 1$,*

$$\Phi_m(x, n, p^{\alpha}) = \frac{x^{mn+1}}{mn+1} \, \frac{p-1}{p^{\alpha-1}(p^{m+1}-1)} \, \frac{1}{D_{(m+1)}} + \Delta\Phi_m(x, n, p^{\alpha}),$$

where $D_{(m+1)}$ is defined as in the preceding theorem, and

$$|\Delta\Phi_m(x, n, p^{\alpha})| \leq A x^{mn} \log x,$$

A being finite and independent of x, m and n.

By Theorem XII of the preceding chapter we have

$$\Phi_m(x, n, p^{\alpha}) = p^{m(\alpha n-1)}(p-1) \sum_{j=0}^{l} p^{m(n-1)j} \Phi_m\left(\frac{x}{p^{\alpha+j}}, n, 1\right),$$

where l is the first value of j for which $\left[\frac{x}{p^{\alpha+j}}\right] = 0$. Put in this expression the value of $\Phi_m(x, n, 1)$ obtained in the preceding theorem. We may write the result equal to $M + N$, where

$$M = \frac{(p-1 x^{mn+1} D_{(m+1)}}{(mn+1) p^{\alpha+m}} \sum_{j=0}^{l} \frac{1}{p^{(m+1)}};$$

and
$$N = p^{m(\alpha n-1)}(p-1)\sum_{j=0}^{l} p^{m(n-1)j}\,\Delta\Phi_m\left(\frac{x}{p^{\alpha+j}}, n, 1\right).$$

We have then
$$|N| \leqq A_1\frac{(p-1)}{p^m}x^{mn}\log x\sum_{j=0}^{l}\frac{1}{p^{jm}},$$
$$\leqq A_2 x^{mn}\log x,$$

where A_2 is finite and independent of x, m and n. Also we may write
$$M = P + Q,$$

where
$$P = \frac{(p-1)}{p^{\alpha+m}}\,\frac{x^{mn+1}}{mn+1}\,\frac{1}{D_{(m+1)}}\sum_{j=0}^{\infty}\frac{1}{p^{(m+1)j}}.$$

and
$$Q = -\frac{p-1}{p^{\alpha+m}}\,\frac{x^{mn+1}}{mn+1}\,\frac{1}{D_{(m+1)}}\sum_{j=l+1}^{\infty}\frac{1}{p^{(m+1)j}}.$$

Now
$$\sum_{j=l+1}^{\infty}\frac{1}{p^{(m+1)j}} = \frac{1}{p^{(m+1)l}}\cdot\frac{1}{p^{m+1}-1},$$

and from our definition of l,
$$p^{\alpha+l+1} > x.$$

Since p occurs to a higher power than this in the denominator on the right in the equation for Q, we may write
$$|Q| \leqq A_3 x^{mn} \leqq A_3 x^{mn}\log x,$$

where A_3 is finite and independent of x, m and n.

Finally, since
$$P = \frac{x^{mn+1}}{mn+1}\,\frac{1}{D_{(m+1)}}\,\frac{p-1}{p^{\alpha-1}(p^{m+1}-1)},$$
the theorem is proved.

We will now derive a formula of approximation for $\Phi_m(x, n, k)$. For shortness of expression, we define a function $P(m, k)$ by the equation
$$P_{(m,k)} = \prod_{i=1}^{r}\frac{p_i-1}{p_i^{\alpha_i-1}(p_i^{m+1}-1)},$$

where $k = \prod_{i=1}^{r} p_i^{\alpha_i}$, and where also we understand
$$P_{(m,1)} = 1.$$

We denote also, as in the preceding theorems, by $D_{(k)}$ the sum $\sum_{i=1}^{\infty}\frac{1}{i^k}$.

We prove now the general theorem:

THEOREM VI.

$$\Phi_m(x, n, k) = \frac{x^{mn+1}}{mn+1}\,\frac{P_{(m,\,k)}}{D_{(m+1)}} + \Delta\Phi_m(x, n, k),$$

where

$$|\Delta\Phi_m(x, n, k)| \leqq Ax^{mn}\log x,$$

A being finite and independent of x, m and n.

Suppose that the theorem holds for $\Phi_m(x, n, k')$, where $k' = \frac{k}{p_r^{\alpha_r}}$, k being equal to $\prod_{i=1}^{r} p_i^{\alpha_i}$.

We have, by Theorem XII of the preceding chapter,

$$\Phi_m(x, n, k) = p_r^{m(\alpha_r n-1)}(p_r-1)\sum_{j=0}^{l} p_r^{m(n-1)j}\,\Phi_m\left(\frac{x}{p_r^{\alpha_r+j}}, n, k'\right),$$

where l is the first value of j for which $\left[\frac{x}{p_r^{\alpha_r+j}}\right] = 0$. By the hypothesis we may write this equal to $M + N$, where

$$M = \frac{p_r-1}{p_r^{m+\alpha_r}}\,\frac{x^{mn+1}}{mn+1}\,\frac{P_{(m,\,k')}}{D_{(m+1)}}\sum_{j=0}^{l}\frac{1}{p_r^{(m+1)j}},$$

and

$$N = p_r^{m(\alpha_r n-1)}(p_r-1)\sum_{j=0}^{l} p_r^{m(n-1)j}\,\Delta\Phi\left(\frac{x}{p_r^{\alpha_r+j}}, n, k'\right).$$

We may, then, proceed exactly as in the preceding theorem, and the result shows that if the theorem is true for k' it is true for k. But in the preceding theorem we have proved it for $k = p^{\alpha}$. It is, therefore, true in general.

For the particular case of the ordinary totient, we have $m = n = k = 1$, and $P(mk) = 1$, while $D_{(2)} = \frac{\pi^2}{6}$. Our formula gives, therefore,

$$\frac{3}{\pi^2}x^2 + \Delta,$$

where $|\Delta| \leqq Ax\log x$; a well-known result (Dirichlet, Werke, vol. II, p. 60; Mertens, Crelle's Journal, vol. LXXVII, p. 289, 1874).

The above results may be looked upon as theorems in connection with the function $\Omega_m(x, n, k)$, where n is negative or zero. The case $n = 1$ is important for the sequel, and a treatment precisely similar to the one used above will give

THEOREM VII.

$$\Omega_m(x, 1, k) = x\frac{P_{(m,k)}}{D_{(m+1)}} + \Delta_{\Omega m}(x, 1, k),$$

where $$|\Delta_{\Omega m}(x, 1, k)| \leq A \log x,$$

A being finite and independent of x, m and n.

This is seen to be the same as would be obtained by the formula for $\Phi_m(x, 0, k)$.

The case $\Omega_1(x, 2, k)$ must be treated separately, since in that case $S\left(\frac{x}{i}, -m(n-1)\right)$ becomes $S\left(\frac{x}{i}, -1\right)$, and the formula of Theorem II must be employed. We start with Theorem X of the preceding chapter and write

$$\Omega_1(x, 2, 1) = \sum_{i=1}^{[x]} \frac{\mu(i)}{i^2} S\left(\frac{x}{i}, -1\right);$$

which gives $\Omega_1(x, 2, 1) = M + N$, where

$$M = \sum_{j=1}^{[x]} \frac{\mu(j)}{j^2} \log\left(\frac{x}{j}\right),$$

and $$N = A\sum_{j=1}^{[x]} \frac{\mu(j)}{j^2}.$$

Therefore, $$|N| \leq A\frac{6}{\pi^2} \leq A_1,$$

A_1 being finite and independent of x.

We also write $$M = P + Q,$$

where $$P = \sum_{j=1}^{[x]} \frac{\mu(j)}{j^2} \log x,$$

and $$|Q| \leq \sum_{j=1}^{[x]} \frac{\log j}{j^2}.$$

Now $\log j < j^{\frac{1}{2}}$, so

$$|Q| \leqq \sum_{j=1}^{\infty} \frac{1}{j^{1+\frac{1}{2}}}\,.$$

which is a finite series (cf. Dirichlet-Dedekind, "Zahlentheorie," p. 304).

Finally, we put $P = R + S$, where

$$R = \log x \sum_{j=1}^{\infty} \frac{\mu(j)}{j^2} = \frac{\log x}{D_{(2)}},$$

and

$$|S| \leqq \log x \sum_{j=[x]+1}^{\infty} \frac{1}{j(j-1)} \leqq \frac{\log x}{[x]} < 1\,.$$

Collecting results, we have

$$\Omega_1(x,\ 2,\ 1) = \frac{\log x}{D_2} + \Delta_{\Omega_1(x,\,2,\,1)},$$

where

$$|\Delta_{\Omega_1(x,\,2,\,1)}| \leqq A,$$

A being finite and independent of x. From this point the discussion runs parallel to the discussion of Theorems V and VI. We obtain thus the theorem:

THEOREM VIII.

$$\Omega_1(x,\ 2,\ k) = \log x\,\frac{P_{(1,\,k)}}{D_{(2)}} + \Delta_{\Omega_1(x,\,2,\,k)},$$

where $|\Delta_{\Omega_1(x,\,2,\,k)}| \leqq A$, *where A is finite and independent of x.*

The remaining values of μ are now readily disposed of. We can prove the theorem:

THEOREM IX. *For* $n > 2$, $x \geqq 1$,

$$\Omega_m(x,\ n,\ k) = \frac{D_{(m.\overline{n-1})}}{D_{(mn)}} + \Delta_{\Omega_m(x,\,n,\,k)},$$

where $|\Delta\Omega_m(x,\ n,\ k)| \leqq \dfrac{A}{[x]}$, *where A is finite and independent of x.*

We start with Theorem X of the previous chapter and write

$$\Omega_m(x,\ n,\ 1) = \sum_{j=1}^{[x]} \frac{\mu(j)}{j^{mn}}\, S\left(\frac{x}{j},\ -m(n-1)\right),$$

and, by Theorem III of this chapter, we write this equal to $M + N$, where

$$M = D_{(m.\overline{n-1})} \sum_{j=1}^{[x]} \frac{\mu(j)}{j^{mn}}\,,$$

and

$$N=\sum_{j=1}^{[x]} \frac{\mu(j)}{j^{mn}} \Delta \left(\frac{x}{j} - m(n-1)\right),$$

and by that same theorem,

$$|N| \leqq \sum_{j=1}^{[x]} \frac{1}{j^{mn}} \div \left[\frac{x}{j}\right].$$

This we may write

$$|N| \leqq \sum_{j=1}^{[x]} \frac{1}{j^2\left[\frac{x}{j}\right]}, \quad \leqq \left[\frac{1}{x}\right]^2 + \sum_{j=1}^{[x]-1} \frac{1}{j^2\left(\frac{x}{j}-1\right)}.$$

Now the expression $j^2\left(\frac{x}{j}-1\right)$ is a minimum when $j=\frac{x}{2}$. Giving it this value

$$|N| \leqq \frac{1}{[x^2]} + \frac{[x]-1}{\frac{x^2}{4}}$$

$$< \frac{A_1}{[x]},$$

A_1 being independent of x, m and n.

Also, $M = P + Q$, where

$$P = D_{(m.\overline{n-1})} \sum_{j=1}^{\infty} \frac{\mu(j)}{j^{mn}},$$

$$= \frac{D_{(m.\overline{n-1})}}{D_{(mn)}}:$$

and

$$|Q| \leqq D_{(m.\overline{n-1})} \sum_{j=[x]+1}^{\infty} \frac{1}{j^{mn}}$$

$$\leqq \frac{A_2}{[x]},$$

where A_2 is again finite and independent of x, m and n. The theorem is thus proved for $k=1$. The proof then proceeds as before.

CHAPTER III.

TOTIENT POINTS.

We define a totient point in space of m dimensions, as a point whose m coordinates are integers having unity for their greatest common divisor. Not to restrict ourselves to positive or non-zero values of the coordinates, we define the greatest common divisor of any set of positive or negative integers, as the greatest common divisor of their absolute values, while the greatest common divisor of any number and zero is the number itself.

The existence of any one totient point with m coordinates involves the existence of $m!$ other totient points, obtained by permuting the coordinates of the point in all possible ways. These points may or may not all be distinct. This is a special case under the more general theorem which follows:

THEOREM I. *The effect of a linear homogeneous substitution with positive or negative integer coefficients and determinant positive or negative unity, is to transform totient points into totient points.*

From the equations of transformation it is manifest that any common divisor of the old coordinates must appear in the new. Solving for the old coordinates in terms of the new, we get again integer coefficients; and again, any common divisor of the new coordinates must appear also in the old. The theorem follows.

We define now an i-compartment of space of m dimensions, as the locus of points which are such that the i^{th} coordinate, x_i, of each is a fixed positive or negative integer, not zero, and if x_j is any other of the $m-1$ coordinates, $\left[\frac{x_j}{x_i}\right]$ is a definite fixed positive or negative integer (or zero).

Any given point lies in m different compartments, since the compartment may be taken with respect to any one of the m coordinates. We obtain an infinitude of compartments for each coordinate x_i by choosing different values of $\left[\frac{x_j}{x_i}\right]$.

THEOREM II. *There exists a one-to-one correspondence between the totient points of any two compartments, obtained by choosing different values of* $\left[\frac{x_j}{x_i}\right]$, *both compartments being taken with respect to the same coordinate* x_i.

By addition of suitable multiples of x_i to the remaining coordinates of a totient point in one compartment, we derive a totient point in the other. But we derive only one. For if any coordinate x_j go by this means to two coordinates x_j' and x_j'', we may write

$$x_j + \lambda_1 x_i = x_j',$$
$$x_j + \lambda_2 x_i = x_j'',$$

whence

$$\left[\frac{x_j}{x_i}+\lambda_1\right]=\left[\frac{x_j'}{x_i}\right],$$

and

$$\left[\frac{x_j}{x_i}+\lambda_2\right]=\left[\frac{x_j''}{x_i}\right].$$

But if x_j' and x_j'' belong to points in the same compartment,

$$\left[\frac{x_j'}{x_i}\right]=\left[\frac{x_i''}{x_i}\right],$$

and it follows easily that $\lambda_1=\lambda_2$, and that the coordinates are the same.

THEOREM III. *The number of totient points in any compartment with respect to the coordinate x_i is $\phi_{m-1}(x_i)$.*

It will be remembered $\phi_{m-1}(x_i)$ indicates the number of sets

$$x_i,\ x_1,\ x_2,\ \ldots\ldots,\ x_{i-1},\ x_{i+1}\ \ldots\ldots\ x_m,$$

where $x_i \geqq x_j \geqq 1$, for $j=1,\ 2,\ \ldots\ldots\ m$, and $j \neq i$, and where also

$$[x_i,\ x_1,\ x_2,\ \ldots\ldots,\ x_{i-1},\ x_{i+1},\ \ldots\ldots\ x_m]=1.$$

If, then, whenever any coordinate x_j is equal to x_i, we subtract x_i, thus reducing that coordinate to zero, we get a set satisfying the conditions

$$x_i > x_j \geqq 0, \qquad j=1,\ 2,\ \ldots\ldots\ m \text{ and } j \neq i,$$

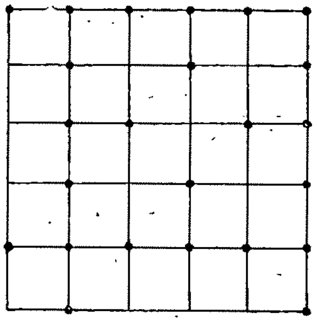
and also still

$$[x_i,\ x_1,\ x_2,\ \ldots\ldots,\ x_{i-1},\ x_{i+1},\ \ldots\ldots,\ x_m]=1.$$

We have, then, exactly $\phi_{m-1}(x_i)$ totient points where the coordinate x_i is the same in each, and where $\left[\frac{x_j}{x_i}\right]=0$. All these points have positive or zero coordinates. They all lie in the same compartment, which, for convenience, we may call the zero compartment. Since the number is the same for each compartment, the theorem follows.

Example I. Take $m=2$, $x_i=6$. Then $\phi_{m-1}(x_i)=2$. The compartments lie on the line $x=6$. In the zero compartment are the points (6, 1) and (6, 5).

Example II. Take $m=3$, $x_i=6$. Then $\phi_{m-1}(x_i)=24$. The compartments lie in the plane $x=6$. In the zero compartment the points are arranged as in the figure:

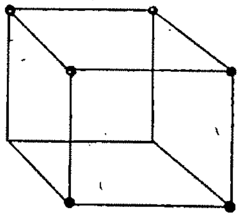


The points in the other compartments are arranged in precisely the same symmetrical way. One obtains the points in another compartment by adding multiples of six to the coordinates y and z of the points in the zero compartment.

Example III. Take $m = 4$, $x_i = 2$ Then $\phi_{m-1}(x_i) = 7$. The compartments are cubes. The zero compartment contains the seven points:

$$(1,\,1,\,1),\ (0,\,1,\,1),\ (1,\,0,\,1),\ (1,\,1,\,0),\ (0,\,0,\,1),\ (0,\,1,\,0),\ (1,\,0,\,0),$$

being the vertices of the unit-cube, with the omission of the corner (0, 0, 0).



For higher values of m, of course, geometrical illustration fails us.

CHAPTER IV.

TOTIENT POINTS IN THE PLANE. APPLICATIONS.

The theory of totient points in space of two dimensions is itself so extensive and furnishes such a variety of applications that we devote a separate chapter to it.

The mere fact, as indicated by the well-known equation

$$\lim_{N=\infty} \frac{\sum\limits_{x=1}^{N} \phi_1(x)}{\frac{1}{2}x^2} = \frac{6}{\pi^2},$$

that the number of totient points in the triangle bounded by the lines $y=0$, $x=y$, $x=N$, is proportional to the area of that triangle, proves nothing as to the uniformity of distribution of such points in the plane. In this chapter we propose to prove this uniformity in the sense that if $N(k)$ is the number of points in or on the boundary of a region of area k, then

$$\lim_{k=\infty} \frac{N(k)}{K} = \frac{6}{\pi^2},$$

the area K increasing in a sense to be explained and the region to be characterized more definitely.

LEMMA. *Let* $\phi_1(x, k)$ *denote the number of integers less than or equal to* $[k]$ *and prime to* x, *then*

$$\phi_1(x, k) = \frac{k}{x} \phi_1(x) + \Delta_{\phi_1(x, k)},$$

where

$$|\Delta_{\phi_1(x, k)}| \leqq x^{\log 2}.$$

For $k > 0$ we know that

$$\phi_1(x, k) = \sum_{(d)} \mu(d) \left[\frac{k}{d}\right],$$

where the sum extends over all d's which are divisors of x. Put

$$\left[\frac{k}{d}\right] = \frac{k}{d} - \sigma_{(k, d)},$$

where $0 \leqq \sigma_{(k, d)} < 1$.

We have then

$$\phi_1(x, k) = k \sum_{(d)} \frac{\mu(d)}{d} + \Delta_{\phi_1(x, k)},$$

where $\Delta\phi_1(x, k) = -\sum \sigma_{(k, d)} \mu(d)$. Suppose $x = \prod_{i=1}^{r} p_i^{\alpha_i}$. Since now $\mu(d) = 0$, when d contains a square factor, we have $\Delta_{\phi_1(x, k)} = -\sum_{(d')} \sigma_{(k, d')} \mu(d')$, where the sum now extends over all d''s which divide $x' = \prod_{i=1}^{r} p_i$. Give now σ and μ their maximum value, unity; then

$$|\Delta\phi_1(x, k)| \leqq 2^r,$$

2^r being the number of divisors of x'. Now, unless x is 1, 2 or 6, we have $x > e^r$, where e is the Naperian base, or $r \leqq \log x$. With these exceptions, therefore,

$$|\Delta\phi_1(x, k)| \leqq 2^{\log x} \leqq x^{\log 2}.$$

But if
$$\begin{aligned} x &= 1,\ \Delta\phi_1(1, k) = 0, && < 1^{\log 2}, \\ x &= 2,\ \Delta\phi_1(2, k) = 1 \text{ or } 0 && < 2^{\log 2}, \\ x &= 6,\ \Delta\phi_1(6, k) = 0,\ 1 \text{ or } 2 && < 6^{\log 2}, \end{aligned}$$

and so the lemma is completely established.

The same proof applies to totient points with negative ordinates. The lemma shows to what degree of approximation the number of totient points on any line is proportional to the length of that line.

It is important for certain applications to generalize the problem before us in that we subject the points to certain conditions. We discuss first the following problem:

PROBLEM. *To find for every real number a, and every angle α $\left(0 < \alpha < \frac{\pi}{2}\right)$, the number, $N(a, \alpha)$, of pairs of integers, (x, y), which satisfy the following conditions*:

$$\begin{gathered} [x, y] = 1, \\ x \equiv 0 \pmod{k}, \\ \frac{y}{x} \leq \tan\alpha, \\ x \leq a. \end{gathered}$$

We are seeking, in fact, the number of totient points whose abscissæ are multiples of a given number k, and which lie in or on the boundary of a triangle AOB whose angle at the origin O is α, and whose side OA lies in the direction of the axis of x, the angle α being acute.

The number of totient points on any ordinate y whose abscissa is kx, is the number of integers less than or equal to y and prime to kx, which, by our lemma, is

$$y\,\frac{\phi_1(kx)}{kx} + \Delta_{\phi_1(kx,\, y)},$$

where
$$|\Delta\phi_1(kx, y)| \leq (kx)^{\log 2}.$$

The number of points in question is then

$$N(a, \alpha) = \sum_{x=1}^{\left[\frac{a}{k}\right]} y\,\frac{\phi_1(kx)}{kx} + \Delta\phi_1(kx, y),$$

But the length of any ordinate y is $kx \tan \alpha$, so

$$\begin{aligned} N(a,\, \alpha) &= \sum_{x=1}^{\left[\frac{a}{k}\right]} \tan \alpha\, \phi_1(kx) + \Delta\phi_1(kx,\, y), \\ &= A + B, \end{aligned}$$

where

$$A = \sum_{x=1}^{\left[\frac{a}{k}\right]} \tan \alpha\, \phi_1(kx),$$

and

$$B = \sum_{x=1}^{\left[\frac{a}{k}\right]} \Delta\phi_1(kx,\, y).$$

By our lemma

$$\begin{aligned} |B| &\leqq \sum_{x=1}^{\left[\frac{a}{k}\right]} (kx)^{\log 2}, \\ &\leqq \sum_{x=1}^{a} x^{\log 2}, \\ &\leqq A_1\, a^{\log 2 + 1}, \end{aligned}$$

where A_1 is finite and independent of a (Theorem I, Chapter II). Now the sum A is nothing less than $\Phi_1(a,\, 1,\, k) \tan \alpha$, and by Theorem VI, Chapter II, this is equal to

$$\frac{6}{\pi^2}\, \frac{a^2}{2} \tan \alpha\, P_{(1,\, k)} + \tan \alpha\, \Delta\Phi_1(a,\, 1,\, k),$$

where $|\Delta\Phi_1(a,\, 1,\, k)| \leqq A_2\, a \log a$. A_2 being finite and independent of a.

Now, $\frac{a^2}{2} \tan \alpha$ is equal to the area, K, of the triangle OAB. Putting together the above results, we have

$$N(a,\, \alpha) = \frac{6}{\pi^2} K P_{(1,\, k)} + \Delta N(a,\, \alpha),$$

where

$$|\Delta N(a,\, \alpha)| \leqq A \tan \alpha \log a + B a^{\log 2 + 1},$$

A and B being finite and independent of a and α.

Take, now, another triangle OAB' having the same base $OA = a$, and a smaller central angle α'. We see at once that the number, $N(t)$, of points (x, y) where

$$[x, y] = 1$$

and

$$x = 0 \pmod{k},$$

which lie in the triangle BOB', is given by the equation

$$N(t) = \frac{6}{\pi^2} t P_{(1,k)} + \Delta N(t),$$

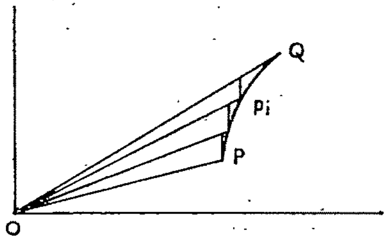
where t is the area of the triangle BOB', and

$$|\Delta N(t)| \leq A \tan \alpha \, a \log a + B a^{\log 2 + 1};$$

A and B being independent of a and α.

Consider, now, a curve, PQ, whose polar equation is

$$r = f(\theta),$$



f being a single-valued, continuous function of θ, for

$$0 \leq \alpha_0 \leq \theta \leq \alpha_1 < \frac{\pi}{2}.$$

We suppose, to avoid unimportant exceptions, that PQ is not a straight line through the origin, nor made up in part of such lines.

Join P and Q to the origin O and call the area $POQ = K$ and the angle $POQ = \beta$. Suppose, further, that the line OQ makes an angle α with Ox, where $\alpha \leq \alpha_1$; and, for simplicity, suppose the figure OPQ to lie in the first quadrant. Divide β into n equal angles θ_n, so that $\theta_n = \frac{\beta}{n}$. Let the separating lines of these angles cut PQ in the $n = 1$ points, p_i, and through these points draw parallels to the y axis. We thus get n triangles t_j ($j = 1 \ldots . n$).

Let $K_{(n)} = \sum_{j=1}^{n} t_j$ be the sum of the areas of these triangles. By choosing n large enough—say n_ϵ—we may make

$$\left| \frac{K - K_{n_\epsilon}}{K} \right| = \eta_{(n_\epsilon)},$$

where $\eta_{(n_\epsilon)} < \frac{\varepsilon}{2}$ where ε is an arbitrarily small positive number taken in advance.

By what precedes, the number of points (x, y) such that $[x, y] = 1$, and $x \equiv 0 \pmod{k}$, which lie in or on the boundaries of the triangles t_j, will be

$$N(K_{n_\epsilon}) = \frac{6}{\pi^2} P_{(1,k)} K_{(n_\epsilon)} + \Delta N(K_{(n_\epsilon)}),$$

where

$$|\Delta N(K_{(n_\epsilon)})| \leqq \sum_{j=1}^{n_\epsilon - 1} A \tan \alpha \, a_j \log a_j + B a_j^{\log 2 + 1},$$

where a_j is the abscissa of p_j. Let R be the largest of these abscissæ, then

$$\begin{aligned} |\Delta N(K_{n_\epsilon})| &\leqq (n_\epsilon - 1)\{A \tan \alpha \, R \log R + B R^{\log 2 + 1}\}, \\ &\leqq \frac{\beta}{\theta_{n_\epsilon}} \{A \tan \alpha \, R \log R + B R^{\log 2 + 1}\}. \end{aligned}$$

Now we have

$$\begin{aligned} K_{n_\epsilon} &= K - (K - K_{n_\epsilon}), \\ &= K - K\eta_{(n_\epsilon)}. \end{aligned}$$

Also

$$N(K_{n_\epsilon}) = N(K) - N(K - K_{n_\epsilon}),$$

and since

$$N(K - K_{n_\epsilon}) < K - K_{n_\epsilon} < \eta_{(n_\epsilon)} K,$$

we have

$$N(K) = \frac{6}{\pi^2} K P_{(1,k)} + \Delta N(K),$$

where

$$|\Delta N(K)| \leqq \frac{\beta}{\theta_{n_\epsilon}} \{A \tan \alpha \, R \log R + B R^{\log 2 + 1}\} + C K \eta_{(n_\epsilon)},$$

where, further,

$$|C| \leqq 1 + \frac{6}{\pi^2} P_{(1,k)} < 2.$$

We have then

$$\left| \frac{\Delta N(K)}{K} \right| \leqq \frac{\beta}{\theta_{(n_\epsilon)}} \left\{ \frac{A \tan \alpha \, R \log R}{K} + \frac{B R^{\log 2 + 1}}{K} \right\} + 2\eta_{(n_\epsilon)}.$$

Multiply, now, every ordinate and every abscissa by M. This multiplies R by M and K by M^2. It leaves β, α, $\theta_{(n_\epsilon)}$ and $\eta_{(n_\epsilon)}$ unchanged. Calling the new area $\overline{K}$, we have

$$\left|\frac{\Delta N(\overline{K})}{\overline{K}}\right| \leqq \frac{\beta}{\theta_{(n_\epsilon)}}\left\{\frac{A \tan\alpha\, R \log RM}{MR} + \frac{BR^{\log 2+1}}{M^{1-\log 2}\,K}\right\} + 2\eta_{(n_\epsilon)}.$$

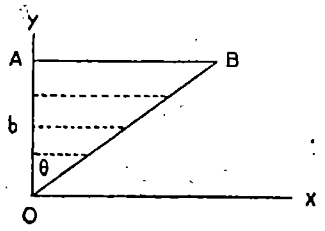
By making M arbitrarily large, we may make this ratio approach as nearly as we please to the quantity $2\eta_{(n_\epsilon)}$, which is less than the arbitrarily small quantity ϵ. The ratio $\dfrac{N(K)}{K}$ approaches, therefore, the limit $\dfrac{6}{\pi^2}P_{(1,\,k)}$, as the area K is increased in the above manner. It is now seen why we restricted α to be less than $\dfrac{\pi}{2}$. Since totient points are symmetric with respect to the axis of x, the curve may cross the x axis. By horizontal summation we can establish the same result for curves crossing the y axis as follows:

Writing $x = mk$, we see that for a totient point it is necessary that $[m, y] = 1$.

1st. Suppose $[k, y] = 1$. On any abscissa of length equal to x, there will be as many totient points as there are values of m such that $[m, y] = 1$, and $m \leqq \dfrac{x}{k}$, which number, as we know, is $\dfrac{x}{k}\,\dfrac{\phi_1(y)}{y} + \Delta\phi_1\left(\dfrac{x}{k}, y\right)$, where $\left|\Delta\phi_1\left(\dfrac{x}{k}, y\right)\right| \leqq y^{\log 2}$.

2nd. Suppose $[k, y] \neq 1$; then, instead of the above result, we should count none at all, since on such an abscissa $[x, y] \neq 1$. Our plan is naturally to effect the sum for all abscissæ under the first supposition, and then correct for those where $[k, y] \neq 1$.

Take our triangle now as in the figure, with base b on the y axis, and let



the angle $AOB=\theta$. Then $x=y\tan\theta$. Under the first supposition the sum is

$$\sum_{y=1}^{[b]}\frac{x}{k}\,\frac{\phi_1(y)}{y}+\sum_{y=1}^{[b]}\Delta\phi_1\left(\frac{x}{k},\,y\right).$$

For the abscissæ where y is a multiple of p_i, one of the factors of k, we have to reject

$$\sum_{y=1}^{\left[\frac{b}{p_i}\right]}\frac{x}{k}\,\frac{\phi_1(p_i y)}{p_i y}+\sum_{y=1}^{\left[\frac{b}{p_i}\right]}\Delta\phi_1\left(\frac{x}{k},\,p_i y\right).$$

By applying the principle of cross-classification, the result is easily seen to be

$$N(t)=\sum_{y=1}^{\left[\frac{b}{d}\right]}\sum_{(d)}\frac{x}{k}\,\mu(d)\,\frac{\phi_1(yd)}{yd}+\Delta\phi_1\left(\frac{x}{k},\,yd\right),$$

where the inner sum runs over all d's which are divisors of k. For each value dy of the argument, $x=dy\tan\theta$, so the result may be written, changing the order of summation,

$$N(t)=\frac{\tan\theta}{k}\sum\mu(d)\sum_{y=1}^{\left[\frac{b}{d}\right]}\phi_1(yd)+\sum_{y=1}^{\left[\frac{b}{d}\right]}\sum_{(d)}\Delta\phi_1\left(\frac{x}{k},\,yd\right),$$

or, in our usual notation,

$$N(t)=\frac{\tan\theta}{k}\sum_{(d)}\mu(d)\,\Phi_1(b,\,1,\,d)+\sum_{y=1}^{\left[\frac{b}{d}\right]}\sum_{(d)}\Delta\phi_1\left(\frac{x}{k},\,yd\right).$$

We then have

$$N(t)=\frac{\tan\theta}{k}\;\frac{6}{\pi^2}\;\frac{b^2}{2}\sum_{(d)}\mu(d)\,P_{(1,\,d)}+\Delta N(t);$$

where $$|\Delta N(t)|\leq A\tan\theta\,b\log b+Bb^{\log 2+1}.$$

Now, it is easily seen that

$$\sum_{(d)}\mu(d)\,P_{(1,\,d)}=\prod_{i=1}^{r}\left(1-P_{(1,\,p_i)}\right),$$

where the product extends over all the primes p_i which divide k. But since

$P_{(1,\,p_i)} = \dfrac{1}{p_i+1}$, the above product may be written

$$\prod_{i=1}^{r} \frac{p_i}{p_i+1},$$

which, when divided by k, is precisely $P_{(1,\,k)}$, and since further $\frac{b^2}{2}\tan\theta = K$, the area of the triangle, we have

$$N(t) = \frac{6}{\pi^2} P_{(1,\,k)} K + \Delta N(t),$$

the limits of Δ being as above. The rest of the argument now proceeds as before. We see incidentally that we would have obtained the same result in the first instance if we had used the condition $y \equiv 0 \pmod{k}$, instead of $x \equiv 0 \pmod{k}$. A finite number of additions and subtractions will not alter the degree of the residue function, and we now may state the theorem:

THEOREM I. *Given a closed contour decomposable into a finite* number of segments of the type considered. Let the area of the region bounded by this contour be K, and let $N(K, \,|k)$ be the number of points (x, y) in or on the boundary such that*

$$[xy] = 1,$$
$$x \equiv 0 \pmod{k},$$

then

$$\lim_{k=\infty} \frac{N(K, \,|k)}{K} = \frac{6}{\pi^2} P_{(1,\,k)},$$

where K increases by ordinary magnification, all the lines of the figure being lengthened proportionally, and where for $k = \prod_{i=1}^{r} p_i^{a_i}$,

$$P_{(1,\,k)} = \frac{1}{k} \prod_{i=1}^{r} \frac{p_i}{p_i+1}, \text{ but } P_{(1,\,1)} = 1.$$

The condition, $x \equiv 0 \pmod{k}$, may, as we have seen, be replaced by the conditions $y \equiv 0 \pmod{k}$, the limit remaining as before. Also, the second part of the above discussion shows that if we assume $x \equiv 0 \pmod{k}$ and $y \equiv 0 \pmod{k'}$—

---

* The theorem might be applied also to curves where a finite decomposition is impossible, provided the total area is a definite thing and the infinite series of residue errors suitably convergent.

where necessarily $[k, k'] = 1$—the result would have been

$$\lim_{k=\infty} \frac{N(K)}{R} = \frac{6}{\pi^2} P_{(1,\, k)} P_{(1,\, k')},$$

which may be written $\frac{6}{\pi^2} P_{(1,\, kk')}$.

It is further possible to impose the conditions $[x, z] = 1$, $[y, z'] = 1$, so that our point (x, y) now shall satisfy the five conditions

$$\begin{array}{rl} \text{I.} & [x, y] = 1, \\ \text{II.} & x \equiv 0 \pmod{k}, \\ \text{III.} & y \equiv 0 \pmod{k'}, \\ \text{IV.} & [x, z] = 1, \\ \text{V.} & [y, z'] = 1, \end{array}$$

where necessarily for totient points to exist,

$$\begin{aligned} [x, k'] &= 1, \\ [k, z] &= 1, \\ [k', z'] &= 1, \end{aligned}$$

but where z and z' may or may not be relative primes.

The number of points satisfying I, II and III is, as we have just seen, $\frac{6}{\pi^2} KP_{(1,\, kk')}$, which, for shortness, we denote by M. Suppose, now, that $z = \prod_{i=1}^{\rho} r_i^{\alpha_i}$ and $z' = \prod_{i=1}^{\sigma} s_i^{\beta_i}$, where r_i and s_j are not necessarily distinct. We reject first those values of x which are multiples of the primes r_i. By applying the principle of cross-classification, the remaining number of points is $M \prod_{i=1}^{\rho} (1 - P(1, r_i))$, which equals $M \prod_{i=1}^{\rho} \frac{r_i}{r_i + 1}$, which equals $MzP_{(1,\, z)}$. If now we suppose $[z, z'] = 1$, we reject in the same way those points where y is a multiple of s_i, and get $MzP_{(1,\, z)} \cdot z' \cdot P_{(1,\, z')}$, which is the desired result when $[z, z'] = 1$. Suppose, however, in addition that x and y are both to be prime to z''. Rejecting those values of x which are multiples of the primes of z'', we have the above result multiplied by $z'' P_{(1,\, z'')}$. From this we need reject now only those points where y is also a multiple of the primes of z''. This multiplies the result again by $\phi_1(z'')$. We may put all these results in the following theorem:

THEOREM II. *Given a closed contour decomposable into a finite number of segments such that for each segment the radius vector is a single-valued, continuous function of the amplitude. Let the area of the region bounded by this contour be* K, *and let* $N(K|, k, k', z, z', z'')$ *be the number of points* (x, y) *such that*

$$\begin{array}{rl} \text{I.} & [x, y] = 1, \\ \text{II.} & x \equiv 0 \pmod{k}, \\ \text{III.} & y \equiv 0 \pmod{k'}, \\ \text{IV.} & [x, z] = 1, \\ \text{V.} & [y, z'] = 1, \\ \text{VI.} & [xy, z''] = 1, \end{array}$$

where also

$$\begin{array}{rl} \text{VII.} & [k, k'] = 1, \\ \text{VIII.} & [k, z] = 1, \\ \text{IX.} & [k', z'] = 1, \\ \text{X.} & [z, z'] = 1, \end{array}$$

$$\textit{then } \lim_{K=\infty} \frac{N(K|, k, k', z, z', z'')}{K} = \frac{6}{\pi^2} zz' \phi_1(z'') P_{(1,\, kk')} P_{(1,\, zz')} P_{(1,\, zz'')},$$

where K *increases, by ordinary magnification, all the lines of the figure being lengthened in the same proportion, and where, for* $k = \prod_{i=1}^{r} p_i^{\alpha_i}$,

$$P_{(1,\, k)} = \frac{1}{k} \prod_{i=1}^{r} \frac{p_i}{p_i + 1}, \textit{ but } P_{(1,\, 1)} = 1.$$

It is not difficult to show that the restriction $ax \not\equiv by \pmod{k}$ introduces a factor $kP_{(1,\, k)}$.

We have seen that in space of any number of dimensions the effect of a linear transformation of determinant positive or negative unity, with positive or negative integer coefficients, is to throw totient points into totient points. We know, further, that in the plane such a transformation leaves areas unaltered. Let such a transformation be

$$x' = \alpha x + \beta y;$$
$$y' = \gamma x + \delta y;$$

where $\alpha\delta - \beta\gamma = 1$. The ten conditions of the preceding theorem are now, dropping accents,

$$\begin{aligned}
&\text{I.} && [x, y] = 1,\\
&\text{II.} && \delta x \equiv \beta y \pmod{k},\\
&\text{III.} && \gamma x \equiv \alpha y \pmod{k'},\\
&\text{IV.} && [\delta x - \beta y, z] = 1,\\
&\text{V.} && [\gamma x - \alpha y, z'] = 1,\\
&\text{VI.} && [(\delta x - \beta y)(\gamma x - \alpha y), z''] = 1.
\end{aligned}$$

The remaining conditions are not changed. The formula for $\lim\limits_{k=\infty} \dfrac{N(K|, k, k', z, z', z'')}{K}$ is not disturbed.

We are now prepared to write out an indefinite number of theorems which are merely applications of the preceding theorems. It has been noticed (cf. Sylvester, Philosophical Magazine, 1883, p. 251), that the number of proper fractions in their lowest terms whose denominators are less than or equal to n, is approximately $\frac{3}{\pi^2} n^2$. This follows easily when thrown into the language of our theorems. We are, as a matter of fact, finding the number of pairs of integers, $[x, y]$, such that $[x, y] = 1$, and also such that $1 \leqq y \leqq x \leqq n$. Here we have the area $K = \frac{n^2}{2}$ and the number is, therefore, $\frac{6}{\pi^2} \frac{n^2}{2}$ or $\frac{3}{\pi^2} n^2$. More generally we might ask for the number of such fractions, where the numerators lie between l and $l + m$, while the denominators lie between l' and $l' + m'$. The area K is here mm', and the number in question is $\frac{6mm'}{\pi^2}$. Manifestly, the number of such theorems may be multiplied indefinitely, the one difficulty in any case being the determination of the area K which stands for the conditions of inequality.

Another class of theorems has to do with integral right triangles. By an integral right triangle we mean a right triangle whose three sides may be represented by integers. Such a triangle is said to be reduced if the three numbers which represent the sides have no common divisor except unity. It is well known* that the three sides of such a triangle are given by the formulæ

---

* See Frenicle, "Traité des triangles rectangles en nombres," Paris, 1676, §§ xxiv, xxv, pp. 59, 61; Euler, "Commentationes arithmeticæ, vol. I, pp. 24, 25. Also Annals of Mathematics, vol. I, 2d series, No. 3, p. 1.

$$a = m^2 + n^2,$$
$$b = m^2 - n^2,$$
$$c = 2mn.$$

If the triangle is to be reduced, it is further necessary and sufficient that $[m, n] = 1$, and $m \not\equiv n \pmod 2$. If we take the further condition that the hypotenuse a shall be less than or equal to N, we have $m^2 + n^2 \leqq N$. If the sides are to be positive, we take $m > n$. We may take m and n both positive, since $-m$ and $-n$ give the same triangles as $+m$ and $+n$. These conditions give an area K equal to $\frac{N\pi}{8}$. Our number, therefore, is $\frac{6}{\pi^2}\,\frac{N\pi}{8}\,2P_{(1,2)}$ or $\frac{N}{2\pi}$. Thus for $N = 100$, $\frac{N}{2\pi} = 15.9$, and actual count gives 16 triangles.

If, instead of restricting the hypotenuse as above, we restrict the sum of the three sides to be less than or equal to N, we easily get the formula $\frac{N}{\pi^2}\log 2$. Manifestly, here also the number of special problems may be indefinitely extended.

The foregoing problem is of special importance in that it suggests a class of theorems of which it is a very special case. For any number to serve as the largest side of such a triangle, it is necessary and sufficient that it should be expressible as the sum of two squares which are relative primes. But the necessary and sufficient condition that it be so expressible is that every one of its prime divisors be of the form $4n + 1$. Further, if the order of the squares be left out of account, a number x of this sort can be so expressed in $2^{\nu(x)-1}$ different ways, where $\nu(x)$ is the number of distinct prime divisors of x. We have, then, the theorem

$$\lim_{N=\infty} \frac{\sum\limits_{x=1}^{N} 2^{\nu(x)}\,\Theta_{(4,1)}(x)}{N} = \frac{1}{\pi}.$$

where the function $\Theta_{(4,1)}(x)$ is equal to 1 or 0, according as all the prime factors of x are or are not of the form of $4n + 1$. This is, then, a particular case of the theorem noted in the Introduction.

We now proceed to consider representations of numbers by the binary quadratic form

$$ax^2 + 2bxy + cy^2,$$

where $[a, 2b, c] = 1$, these representations to conform to the three conditions

$$\begin{aligned} &\text{I.} \quad [x, y] = 1, \\ &\text{II.} \quad 0 < ax^2 + 2bxy + cy^2 \leqq N, \\ &\text{III.} \quad [ax^2 + 2bxy + cy^2, 2D] = 1, \end{aligned}$$

where $D = b^2 - ac$.

We may suppose $[a, 2D] = 1$ (Dirichlet-Dedekind "Zahlentheorie," p. 233), and, therefore, $[a, b] = 1$. Now we have

$$a(ax^2 + 2bxy + cy^2) = (ax + by)^2 - Dy^2.$$

We may, therefore, replace condition III by

$$[ax + by, 2D] = 1.$$

Our result is, therefore, $\frac{6}{\pi^2} K 2D P_{(1, 2D)}$. The area K is to be determined from the second condition, and is very different in form according as D is positive or negative; that is, for Definite or Indefinite forms.

For Definite forms, where D is negative and equal to $-\Delta$, the area is bounded by the ellipse

$$ax^2 + 2bxy + cy^2 = N.$$

We have, then, $K = \frac{\pi N}{\sqrt{\Delta}}$, and the number of points in this case satisfying the given conditions is

$$\frac{12}{\pi} N\sqrt{\Delta} P_{(1, 2\Delta)}.$$

For Indefinite forms, we may have an infinite number of representations of a number m by one and the same form. Thus, if

$$ax^2 + 2bxy + cy^2 = m$$

be a representation of m, so also is

$$a\xi^2 + 2b\xi\eta + c\eta^2 = m,$$

where

$$\begin{aligned} \xi &= U_{(n)} . (bx + cy) \pm T_{(n)} . x, \\ \eta &= U_{(n)} (ax + by) \mp T_{(n)} . y, \end{aligned}$$

where $(T_{(n)}, U_{(n)})$ is any one of the infinite number of solutions of

$$t^2 - Du^2 = 1.$$

Now, it is well known (Dirichlet-Dedekind "Zahlentheorie," p. 247) that one of these solutions may be isolated from the rest by the conditions

$$y > 0,$$
$$U(ax+by) - Ty > 0.$$

These conditions define a hyperbolic sector within which all our points must lie, which is bounded by the lines

$$ax^2 + 2bxy + cy^2 = N,$$
$$y = 0,$$
$$U(ax+by) - Ty = 0,$$

the area of which is

$$\frac{N}{2\sqrt{D}} \log (T + U\sqrt{D}),$$

(T, U) being the fundamental or smallest solution of the Pellian equation $t^2 - Du^2 = 1$. This being our area K, our number of points in this instance is

$$\frac{6}{\pi^2} \sqrt{D}\, P_{(1,\, 2D)}\, N \log (T + U\sqrt{D}).$$

If we multiply the above results by h, the number of properly primitive classes of determinant D, we get the total number of properly primitive representations of numbers less than or equal to N and prime to $2D$. This number is otherwise expressible as follows: Let x be any number prime to $2D$, and let $\nu(x)$ be the number of its distinct prime factors. If D is not a quadratic residue of each one of these prime factors, x is not capable of primitive representation by any form of determinant D. If, however, D is a quadratic residue of every prime factor of x, then the number of primitive representations of x by properly primitive forms of determinant D is $\varepsilon 2^{\nu(x)}$, where ε is the number of solutions of the Pellian equation $t^2 - Du^2 = 1$.

In the case of Definite forms $\varepsilon = 2$, except for the single case $D = -1$, where $\varepsilon = 4$.

In the case of Indefinite forms, the number of solutions of the Pellian equation is infinite, but our isolating conditions noted above amount to making $\varepsilon = 1$.

Now the primes, of which D is a quadratic residue, belong to a certain set s of linear forms (Dirichlet-Dedekind, "Zahlentheorie," p. 121). If x is then made up of primes belonging to these forms, we get $2^{\nu(x)}$ primitive representations, otherwise none at all. If, then, as in the Introduction, we define a function $\Theta_s(x)$, which equals 1 or 0, according as each prime divisor of x does or does not belong to a form of the set s of linear forms, we may write for the above number of properly primitive representations

$$\varepsilon \sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x).$$

For Definite forms this gives

$$\varepsilon \lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N} = \frac{12}{\pi} h \sqrt{\Delta} P_{(1,\,2\Delta)}.$$

For Indefinite forms, we get

$$\varepsilon \lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N} = \frac{6}{\pi^2} h \sqrt{D} P_{(1,\,2D)} \log(T + U\sqrt{D}).$$

We have thus established, for a large number of cases, the theorem mentioned in the Introduction. The method will not avail to establish the law for other systems of forms s, such as for example the single form $4n-1$, where the law seems to be

$$\lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N} = \frac{2}{\pi}.$$

The form $4n-1$ belongs to quadratic forms only in connection with other linear forms.

From these last equations we may derive new expressions for the number h of properly primitive classes of determinant D.

For Definite forms

$$h = \varepsilon \frac{\pi}{12} \frac{1}{\sqrt{\Delta}} P_{(1,\,2\Delta)} \lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N}.$$

For Indefinite forms,

$$h = \frac{\pi^2}{6\sqrt{D}} \cdot \frac{1}{P_{(1,\,2D)} \log(T + U\sqrt{D})} \lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N}.$$

Again, from the well-known formulæ for h (Dirichlet-Dedekind, "Zahlentheorie," §§97–101), we may write the equation for Definite forms,

$$\lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N} = \frac{12}{\pi^2} \Delta P_{(1,\, 2\Delta)} \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{D}{i}\right).$$

where, on the right, the sum (which is not independent of the order of the terms) is arranged according to increasing values of i, the symbol $\left(\frac{D}{i}\right)$ being Jacobi's symbol.

For Indefinite forms, the equation is

$$\lim_{N=\infty} \frac{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_s(x)}{N} = \frac{12}{\pi^2} D P_{(1,\, 2D)} \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{D}{i}\right).$$

From these last two expressions, we observe that if s and s' denote the sets of linear forms belonging to binary quadratic forms of determinants D and $-D$ respectively, we have

$$\lim_{N=\infty} \frac{\sum_{i=1}^{N} 2^{\nu(x)} \Theta_s(x)}{\sum_{x=1}^{N} 2^{\nu(x)} \Theta_{s'}(x)} = \frac{\sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{-D}{i}\right)}{\sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{D}{i}\right)}.$$

Let us restrict ourselves to numbers m_i, which belong to a particular form of the above set s of linear forms. (Clearly the factors of m_i may or may not belong to this particular form.) This restriction might be written as a congruential condition on m_i. It might then be written as a congruential condition on x and y. The modulus of this congruential relation would depend only on the modulus of the forms of the system s, and so would be the same, whatever particular form we have selected. Each linear form of s, therefore, furnishes the same number to the above sum, and the number thus furnished by each is the total number divided by the number of forms in s.

Now, the total number of linear forms belonging to a quadratic form of determinant D is well known. (Cf. H. J. S. Smith, Works, vol. I, pp. 206, 207.) If we write

$$D = 2^a \prod_{i=1}^{r} p_i^a,$$

where the p_i's are odd primes, the number in question is

$$\tfrac{1}{2}\phi_1(2^k D'),$$

where $D' = \prod_{i=1}^{r} p_i$ and k is given as follows:

$k = 1$, when $D \equiv 1$ or 5 (mod 8),
$k = 2$, " $D \equiv 3$, 4 or 7 (mod 8),
$k = 3$, " $D \equiv 0$, 2 or 6 (mod 8).

But $\tfrac{1}{2}\phi_1(2^k D') = 2^{k-2}\prod_{i=1}^{r}(p_i - 1)$, so that we have for each form

$$\frac{12}{\pi}\,\frac{D}{2^{k-2}}\,\frac{P_{(1,\,2D)}}{\prod_{i=1}^{r}(p_i-1)}\,K.$$

This is not a case of the theorem noted in the Introduction. In this we have imposed the further restriction that the number x should belong itself to a particular form, its factors belonging to the forms of s.

CHAPTER V.

FURTHER RESULTS AND DESIDERATA.

The theory of totient points in space of m dimensions is as yet incomplete. Proofs of the following theorems have been obtained, however:

THEOREM I. *Denoting by* $\phi_m(x, k_1, k_2, \ldots, k_m)$ *the number of sets of integers* $x_1 x_2 \ldots x_m$ *such that* $[x, x_1, x_2, \ldots, x_m] = 1$ *and* $x \geqq k_i \geqq x_i \geqq 1$, *we have*

$$\phi_m(x, k_1, \ldots k_m) = \sum_{(d)} \prod_{i=1}^{m} \left[\frac{k}{d}\right] \mu(d),$$

the sum extending over all d's which are divisors of x.

By means of this we get

THEOREM II. $\phi_m(x, k_1, \ldots, k_m) = \prod_{i=1}^{m} k_i \dfrac{\phi_m(x)}{x^m} + \Delta\phi_m(x, k_1, \ldots, k_m)$, *where* $|\Delta\phi_m(x, k_1, \ldots, k_m)| \leqq A\bar{k}^{m-1}x^{\log 2}$, *where A is finite and independent of x, and* $\bar{k}$ *is the largest of the parameters* k_i.

This theorem is seen to be a generalization of the lemma of Chapter IV. By means of it we have established a certain "density theorem" for totient points in space of m dimensions. We use the following notions and definitions:

An m dimensional surface is the locus of points $(x_1, x_2, \ldots, x_m)$ satisfying a single relation $F(x_1, x_2, \ldots, x_m) = 0$. Two points not on the surface will be said to lie on *opposite sides* if, when their coordinates are substituted in $F(x_1, x_2, \ldots x_m)$, the two results are different in sign.

If all the points lying on one side of a surface have all their coordinates finite, the surface will be called *closed*, and the points will be said to be on the *inside*.

The *content* of a closed surface will be defined by the integral

$$V_{(m)} = \int_{x_1} \int_{x_2} \ldots \int_{x_m} dx_1\, dx_2 \ldots dx_m,$$

the limits being taken so as to include all the elements $dx_1 dx_2 \ldots dx_m$ lying on the inside of the surface.

We speak also of the *intersection* of two m dimensional surfaces as the locus of points satisfying the equations of both. Points lying on a definite side of each surface may have all their coordinates finite, in which case we may speak of the content enclosed by the two surfaces.

We have then established the following theorem:

THEOREM III. *The number of totient points within or on any closed surface of m dimensions being denoted by $N(V)$, where V is the content of the surface, we have*

$$\lim_{V=\infty} \frac{N(V)}{V} = \frac{1}{\sum\limits_{i=1}^{\infty} \frac{1}{i^m}},$$

where the content V is supposed to increase by multiplying the coordinates of every point on the surface by the same multiplier.

We have not discussed those cases where the coordinates are subjected to further restrictions. It is hoped that theorems concerning primitive representation by m-ary forms may be obtained, with perhaps applications similar to those obtained in the case of binary quadratic forms.

A method of discussing the following problem is also still lacking:

"To prove or disprove the equation

$$\lim_{N=\infty} \frac{\sum\limits_{x=1}^{N} 2^{\nu(x)} \Theta_{(a,\,b)}(x)}{x} = \text{constant},$$

where $\Theta_{(a,\,b)}(x) = 1$ or 0, according as all the $\nu(x)$ distinct primes in x are or are not of the form $an + b$ where $[a, b] = 1$."

A proof of this theorem, if it is true, would furnish easily a proof of Dirichlet's theorem that the number of primes of the form $an + b$, where $[a, b] = 1$, is infinite. We have, as we have seen, proved the theorem for the forms $4n + 1$ and $6n + 1$, which belong to the forms $x^2 + y^2$ and $x^2 + 3y^2$ respectively. For quadratic forms in general, we have to reckon with more than one linear form.

In connection with this last theorem, we have established the following equation, which may be of assistance in solving the problem:

$$\sum_{x=1}^{[N]} 2^{\nu(x)}\,\Theta_{(a,\,b)}(x) = \sum_{x=1}^{[N]} T\left(\frac{N}{x}\right)\mu(x),$$

where $T(k) = \sum_{j=1}^{[k]} \prod_{i=1}^{r}{}_{(j)}\,(1 + 2\alpha_i)$, where also $j = \prod_{i=1}^{r} p_i^{\alpha_i} \prod_{i=1}^{s} q_i^{\beta_i}$, where, we suppose,

p_i and q_i are primes $\geqq 1$ such that

$$p_i = an + b,$$
$$q_i \neq an + b.$$

Concerning Klein's Group of $(n+1)!$ n-ary Collineations.[*]

By Eliakim Hastings Moore.

---

1. To the statement that a set of $n+1$ real numbers

$$y_0, y_1, \ldots, y_n$$

determines a point, one may give six geometric interpretations (which are really of the nature of definitions), in that the numbers y_i are

(I) Cartesian point coördinates in real flat space R_{n+1} of $n+1$ dimensions;
(II_u, II_b) Homogeneous point coördinates in R_n;
(III) Supernumerary Cartesian point coördinates in R_n;
(IV_u, IV_b) Supernumerary homogeneous point coördinates in R_{n-1};

with the restriction (in II and IV) that not all the homogeneous coördinates y of a point are 0, and (in III and IV) that the sum of the supernumerary coördinates y of a point is 0, and with the understanding that the points

$$(y'_0, \ldots, y'_n), \quad (y''_0, \ldots, y''_n)$$

are the same point if and only if

$$y''_i = \rho y'_i, \qquad (i = 0, 1, \ldots, n),$$

where the proportionality factor ρ is a real number:

(I, III): ρ is 1;
(II_u, IV_u): ρ is any positive number;
(II_b, IV_b): ρ is any non-zero number.

One speaks of the geometries of metric space, of unilateral and of bilateral projective space. The bilateral projective space is the ordinary projective space, while the unilateral projective space R_n is in effect the space of rays diverging from a point in metric space R_{n+1}.

---

[*] Presented to the American Mathematical Society, Chicago section, December 30, 1898, and conveniently modified in the present writing.

The supernumerary coördinates $y_0, \ldots, y_n$ (III, IV) are most conveniently introduced in terms of $x_1, \ldots, x_n$ (I, II) by the formulas:

$$(n+1)\,y_i \equiv (n+1)\,x_i - \sum_{i=0}^{i=n} x_i, \qquad (i=0, 1, \ldots, n),$$

where x_0 is permanently 0, so that in the x's one has the identities:

$$\sum_{i=0}^{i=n} y_i \equiv 0; \quad y_i - y_j \equiv x_i - x_j, \qquad (i, j = 0, 1, \ldots, n).$$

2. Denoting by $a_0, a_1, \ldots, a_n$ any permutation a of the $n+1$ indices $0, 1, \ldots, n$, we consider the collineation S_a which transforms the point $(y_0, \ldots, y_n)$ to the point $(y'_0, \ldots, y'_n)$, where

$$y_i = y'_{a_i}, \qquad (i=0, 1, \ldots, n).$$

The $(n+1)!$ collineations S_a corresponding to the $(n+1)!$ permutations are distinct, and form a group $G_{(n+1)!}$ of collineations simply isomorphic to the symmetric group on $n+1$ letters. For the interpretation IV_b this is Klein's group[*] of collineations permuting in all ways $n+1$ independent points of R_{n-1}.

3. In my paper[†] on the cross-ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions, there are contained general theorems[†] of considerable interest concerning the fixed points (real and complex) of the collineations of Klein's group.

4. It seems desirable to determine, for the six well-known groups $G_{(n+1)!}$ of §2, the corresponding partitions of the respective spaces $(R_{n+1}, R_n, R_n, R_{n-1})$ into $(n+1)!$ regions. The regions are to be simply connected and conjugate or equivalent under the group. And a point P, which is invariant under exactly d collineations, is to belong to exactly d regions which permute transitively under those collineations, and two distinct points of a region are never to be con-

---

[*] Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346-358, 1871).

[†] American Journal of Mathematics, vol. XXII, p. 279.—Cf. §§2 and 5, and the closing paragraphs of the paper.

jugate under the group. Thus every region contains exactly one of every set of conjugate points.

5. As is usual in the geometric group theory (after Klein), we take one of the regions as the fundamental region and denote it by I. Then the collineation S_a throws the region I to a region S_a. If the region I is defined by a system $\sigma(y_0, y_1, \ldots, y_n)$ of conditions (equalities and inequalities) on the coördinates $y_0, y_1, \ldots y_n$ of its general point, it is obvious that the region S_a is defined by the system $\sigma(y_{a_0}, y_{a_1}, \ldots, y_{a_n})$ of conditions on the coördinates $y_0, \ldots, y_n$ of its general point.

6. For the cases I, III, II_u, IV_u one has obviously the following very simple partition:

The region S_a contains all points $(y_0, \ldots, y_n)$ for which

$$y_{a_0} \leqq y_{a_1} \leqq \cdots \leqq y_{a_n}.$$

7. For the cases II_b, IV_b the partition of §6 would assign a point in general to two regions, since the point is unchanged if its coördinates suffer a simultaneous change of sign. By a proper determination of the coördinates as to sign, we may, however, arrange to hold to the partition of §6 also for these cases.

Let a point be given by the various sets $(\rho\eta_0, \ldots, \rho\eta_n)$ of coördinates $(y_0, \ldots, y_n)$, where ρ is any real non-zero number. For the purpose of the partition of §6, we specify that ρ shall have such a sign that when the numbers $\eta_0, \ldots, \eta_n$ are arranged in order* of increasing algebraic magnitude,

$$\zeta_0, \ldots, \zeta_n, \qquad (\zeta_0 \leqq \zeta_1 \leqq \cdots \leqq \zeta_n),$$

the first of the expressions:

$$\zeta_0 + \zeta_n, \ \zeta_1 + \zeta_{n-1}, \ldots, \zeta_m + \zeta_{n-m}, \ldots \qquad (m = 0, 1, \ldots, [n/2]),$$

not zero, shall be negative.† This stipulation, it is important to notice, is inva-

---

*In case of any equalities amongst the η's, this order is not quite definite; this fact does not, however, interfere with the uses made of the order.

†Thus the point

$$(y_0, y_1, \ldots, y_k, \ldots, y_n) = (-n, 1, \ldots, 1, \ldots, 1)$$

is in the fundamental region I.

riant under the collineations of the group. The stipulation fails only if

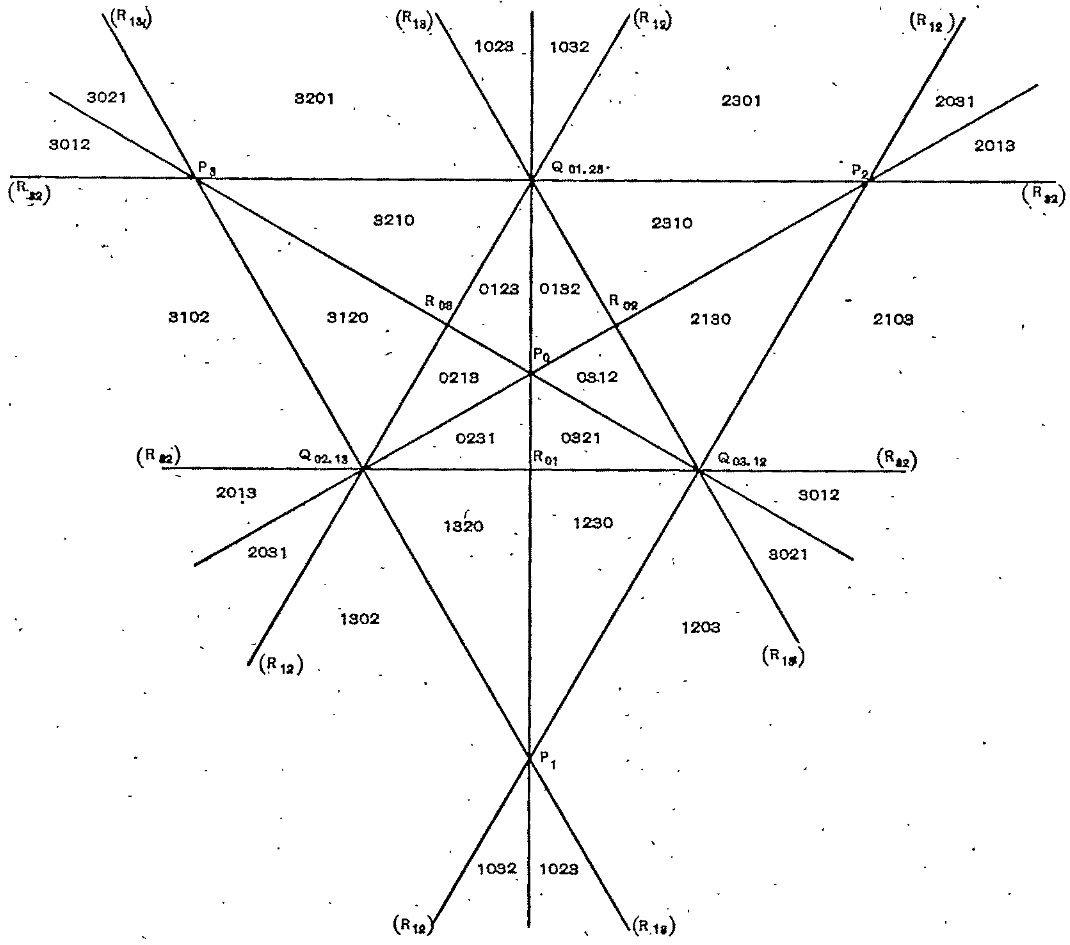
$$0=\zeta_0+\zeta_n=\zeta_1+\zeta_{n-1}=\therefore=\zeta_m+\zeta_{n-m}=\cdot\cdot\,,$$
$$(m=0,1,\cdot\cdot,[n/2]).$$

In such a case, letting ρ have each sign, we assign by §6 the point $(\eta_0,\cdot\cdot,\eta_n)$ to two regions; it is, indeed, invariant under the transformation interchanging the two regions, viz., the transformation interchanging its coördinates

$$\zeta_0,\zeta_n;\ \zeta_1,\zeta_{n-1};\cdot\cdot;\ \zeta_m,\zeta_{n-m};\cdot\cdot,$$

and accordingly it belongs in the two regions.

8. For all cases the partitions which were in view have been obtained. In illustration, I give the figure for the case IV_b: $n=3$. The 4 points $P_0\,P_1\,P_2\,P_3$



are permuted in 4! ways, and the 4! parts of the projective plane have as boundaries the 6 sides of the 4-gon $P_0 P_1 P_2 P_3$, and the 3 sides of its diagonal 3-gon $Q_{01.23}$, $Q_{02.31}$, $Q_{03.12}$. The x-coördinate triangle being $P_1 P_2 P_3$ and the x-unit point being P_0, the y-coördinate 4-side is the 4-side whose 6 vertices are the 6 simple intersections $R_{01}, \ldots, R_{34}$ of the 9 lines of the partition: R_{ij} lies on the lines $P_i P_j$, $Q_{ik.jl}\, Q_{il.jk}$, $y_k = 0$, $y_l = 0$, where $ijkl$ are the 4 indices 0123 in some order. The whole figure is a well-known fundamental figure of projective geometry. A convenient metrical specialization is given here.

The 4! regions $S_a = S_{a_0 a_1 a_2 a_3}$ are marked with the 4! permutations a. The fundamental region I is 0123 with the vertices $P_0 Q_{01.23} R_{03}$. The region $S_{a_0 a_1 a_2 a_3}$ is the triangle $P_{a_0} Q_{a_0 a_1 . a_2 a_3} R_{a_0 a_3}$. The vertices P belong each to 6 regions, the vertices Q belong each to 8 regions, and the vertices R belong each to 4 regions.

The collineations

$$A_2 = S_{0213} = S_{(12)}, \quad A_3 = S_{0132} = S_{(23)}, \quad A = S_{3210} = S_{(03)(12)},$$

constitute a system of generators of the group; they are the projective reflections which throw the region I into the adjacent regions. From the figure in the vicinity of the region I one has the generational relations:

$$(\alpha) \qquad I = A_2^2 = A_3^2 = A^2 = (AA_2)^2 = (AA_3)^4 = (A_2 A_3)^3.$$

These relations, easily verified analytically, are not sufficient fully to characterize the group, for the relation:

$$(\beta) \qquad S_{3102}^3 = S_{(032)}^3 = (A_3 A_2 A)^3 = I,$$

is not derivable from them, since they are all even while it is odd in the elements $A_2 A_3 A$. We come back to this matter* in §10.

---

* The following remarks are of interest. If we introduce

$$A_1 = S_{1023} = S_{(01)} = S_{(03)(12)} S_{(23)} S_{(03)(12)} = AA_3A,$$

we find that in the abstract group $G(A_2, A_3, A)(\alpha) = G(\alpha)$ generated by A_2, A_3, A, subject to the relations (α), the three elements $A_1 = AA_3A$, A_2, A_3 satisfy the standard relations:

$$(\gamma) \qquad I = A_1^2 = A_2^2 = A_3^2 = (A_1 A_2)^3 = (A_1 A_3)^2 = (A_2 A_3)^3$$

for the symmetric group $G_{4!}^{4}$ on 4 letters. (This fact was noticed by Mr. J. C. Hammond.) They generate a subgroup $G(A_1, A_2, A_3) = G_{4!}$ of the group G. That the group generated is not a subgroup of the $G_{4!}$ appears from consideration of the collineation group $G_{4!}$, which is a quotient group of G, and

9. In general the flat spaces

$$y_i - y_j = 0 \qquad (i \neq j;\ i, j = 0, 1, \ldots, n)$$

and, in cases II_b and IV_b,

$$y_i + y_j = 0 \qquad (i \neq j;\ i, j = 0, 1, \ldots, n)$$

effect the partition into the $(n+1)!$ regions. The space $y_i - y_j = 0$ is invariant point by point under the involutoric collineation interchanging the coördinates y_i, y_j—a transformation which reflects the partition into itself on the space, $y_i - y_j = 0$ interchanging the regions of the various pairs of regions lying on opposite sides of that space. The points of $y_i + y_j = 0$, however, apart from those lying in the boundaries just discussed, belong each to one region, with the exception of those satisfying relations of the type

$$0 = y_i + y_j = y_k + y_l = \ldots,$$

as was explained in §7. (Only for $n = 3$ of IV_b is $y_i + y_j = 0$ invariant point by point.) The space $y_i + y_j = 0$ is thus itself divided into regions, each of which belongs to one of the two fundamental regions of which it is a boundary.

10. One might study more closely the topological interrelations of the regions and their boundaries of various dimensions. In particular, by considering the fundamental region I and the regions S_g adjacent to it along boundaries of the highest dimension, one might (as usual[*]) determine a system of generators S_g of the group, and by means of the regions adjacent to it along boundaries of the highest two dimensions, obtain a system of grouptheoretic relations amongst these generators.

---

which is, moreover, generated by A_1, A_2, A_3. On solving in the collineation group $G_{4!}$ for A in terms of A_1, A_2, A_3 one finds

$$A = A_1 A_2 A_3 A_2 A_1 A_2.$$

In this on replacing A_1 by AA_3A one has a relation

$$A = AA_3AA_2A_3A_2AA_3AA_2,$$

which is equivalent under (α) to the relation (β). Thus the group $G(A_2, A_3, A)(\alpha, \beta)$ is its own group $G(A_1, A_2, A_3)$. Hence the relations (α, β) fully characterize the collineation group $G_{4!}$ as an abstract group; and, further, the group $G(\alpha)$ is the direct product of its subgroup $G(A_1, A_2, A_3)$ and the invariant subgroup H whose elements reduce to the identity by the adjunction of the relation (β).

[*] Cf. for example, Klein, "Modulfunctionen," I, p. 452, p. 455. Fricke u. Klein, "Automorphe Functionen," I, p. 168 fg.

With respect to any geometric group with fundamental region, it seems to be clear that the system of relations so obtained is a complete system of generational relations of the group in the sense[†] of Cayley and Dyck in case the fundamental region is simply connected and the set of all the equivalent regions forms a simply connected portion of all space.

These conditions are fulfilled here in cases I, III, and it was indeed in this way, from the geometric symmetric group of case I and its alternating subgroup that I obtained the abstract generational determination of the symmetric and alternating groups on n letters, which, in pure grouptheoretic form, I exhibited in the Proceedings of the London Mathematical Society, Dec. 10, 1896, vol. XXVIII, p. 357. One has the feeling that a generational determination of a group effected by geometric process is apt to be, as in the modular group and in the case just cited, of the utmost simplicity.

The conditions are, however, not fulfilled in cases II, IV, since projective spaces, whether bilateral or unilateral, are not simply connected; a closed straight line which passes through the infinite region of the plane cannot, by continuous deformation, shrink up to a point.—Thus, for example, in the case $\mathrm{IV}_b: n=3$, considered in §8, the broken line corresponding to the relation (β) is closed through infinity.

THE UNIVERSITY OF CHICAGO, *September*, 1900.

[†] Cayley, "The Theory of Groups," American Journal of Mathematics, vol. I, p. 50; Dyck, "Gruppentheoretische Studien," Mathematische Annalen, vol. XX, p. 1. Cf. also my paper: "On the Generational Determination of Abstract Groups," soon to be published in the Transactions of the American Mathematical Society.

The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.[*]

Part First: Geometric Representation.[†]

BY HERBERT ELLSWORTH SLAUGHT.

---

§1.—INTRODUCTION.

1. The group of Cremona transformations, whose geometric representation is the subject of the following paper, is a special case, $n=5$, of the general cross-ratio group of order $n!$.[‡] If these cross-ratio transformations be expressed in homogeneous point coordinates, then each takes the form

$$z_1' : z_2' : z_3' = \phi(z_1, z_2, z_3) : \chi(z_1, z_2, z_3) : \psi(z_1, z_2, z_3).$$

2. The transformations of this system possess the following properties:

(a). They are all algebraic of order

$$\mu \leqq 2.$$

(b). They are birational, those of order $\mu = 2$ having $2^2 - 1$ fundamental points at which the functions ϕ, χ, ψ vanish simultaneously.

(c). If S_i, S_j, S_k are the substitutions on the five indices with which the transformations A_i, A_j, A_k are respectively associated, and if

$$S_i S_j = S_k,$$

---

[*] The study of this group was undertaken as a dissertation at the University of Chicago at the suggestion and under the direction of Professor Moore. A brief abstract of the chief results was printed in Science, July 29, 1898.

[†] In Part Second the complete form-system of invariants of the group is discussed.

[‡] E. H. Moore, "The Cross-Ratio group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (American Journal of Mathematics, vol. XXII, pp. 336–342, 1900).

when compounded from left to right, then

$$A_i A_j = A_k,$$

when compounded from right to left.

3. This system of transformations forms a group distinguished by the following characteristics:

(a). It is holoedrically isomorphic with the symmetric substitution group on five indices.

(b). It is a group of *quadratic Cremona transformations of the plane into itself.*

(c). It has four fundamental points, some three of which belong to every quadratic transformation in the group. See Art. 13.

(d). It is abstractly identical with, and under collineation equivalent to, the Cremona group of order 120 noted by Autonne* and Kantor;† but it is distinguished by the fact that its operators are all *cross-ratio transformations.*‡

4. In studying the geometric representation of this group, the following generational transformations are chosen. Corresponding to the substitutions

$$(34),\ (23)(45),\ (45),\ (15)(34) \text{ and } (12)$$

on the indices 1 .... 5, we find the transformations respectively,

$$\begin{aligned}
K;&\quad z_1' : z_2' : z_3' = z_3 : z_2 : z_1,\\
L;&\quad z_1' : z_2' : z_3' = z_3 - z_2 : z_3 - z_1 : z_3,\\
M;&\quad z_1' : z_2' : z_3' = z_2 : z_1 : z_3,\\
T;&\quad z_1' : z_2' : z_3' = z_3(z_1 - z_2) : (z_1 - z_2)(z_3 - z_2) : z_1(z_3 - z_2),\\
T';&\quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2.
\end{aligned}$$

5. Of these, K, L and M, are collineations and T and T' are quadratic transformations. It is easily seen that all transformations corresponding to permutations on the indices 1 .... 5 which leave 1 fixed are linear, and that all others are quadratic. Hence the 24 linear transformations by themselves form a subgroup $G_{24}^{(1)}$, which, it will be found, may be generated by K, L, M (Art. 7), and then either T or T' will extend $G_{24}^{(1)}$ to the main group G_{120}.

---

* Journal de Mathématiques, series 4, vol. I, 1885, p. 435.

† "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene," p. 105.

‡ E. H. Moore, American Journal of Mathematics, vol. XXII, p. 340, 1900.

§2.—*Geometric Representation of* $G_{24}^{(1)}$.

6. The Klein's[*] linear homogeneous substitution group $G_{4!}$ has been exhibited geometrically[†] in connection with the complete quadrangle (including the diagonals) whose vertices are[†]

$$\left.\begin{array}{ll} O_1; & x_1:x_2:x_3 = 1:1:1, \\ O_2; & x_1:x_2:x_3 = 1:-1:1, \\ O_3; & x_1:x_2:x_3 = 1:1:-1, \\ O_4; & x_1:x_2:x_3 = -1:1:1, \end{array}\right\} \qquad (1)$$

where the boundaries of the fundamental region are defined by[‡]

$$\left.\begin{array}{r} x_3 \geqq 0, \\ x_1 - x_2 \leqq 0, \\ x_1 - x_3 \geqq 0. \end{array}\right\} \qquad (2)$$

And the generators of the group are

$$\left.\begin{array}{ll} K'; & x_1':x_2':x_3' = x_3:x_2:x_1, \\ L'; & x_1':x_2':x_3' = x_1:x_2:-x_3, \\ M'; & x_1':x_2:x_3' = x_2:x_1:x_3. \end{array}\right\} \qquad (3)$$

7. The linear subgroup $G_{24}^{(1)}$ possesses properties similar to those shown by Professor Moore for $G_{4!}$. As a substitution group, it permutes among themselves the four indices 2, 3, 4, 5.

Geometrically, it permutes among themselves certain four points of the plane which may be found by considering the transformations K, L, M set up in Art. 4.

K is a projective reflection on the axis

$$z_1 - z_3 = 0$$

from the center

$$z_1 : z_2 : z_3 = -1 : 0 : 1.$$

L and M are ordinary reflections on the axes respectively,

$$\begin{array}{l} z_1 + z_2 - z_3 = 0, \\ z_1 - z_2 \phantom{{}-z_3} = 0. \end{array}$$

---

[*] Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346–358, 1871).

[†] E. H. Moore, "Concerning Klein's Groups of $(n+1)!$ n-ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 336–342, 1900).

[‡] The fundamental region and the generators are here chosen in a way suitable for convenient use in this paper.

The plot of these lines [Fig. I]* shows that K, L, M permute the 4 points:

$$\left.\begin{array}{ll} Q_2; & z_1:z_2:z_3=1:1:1, \\ Q_3; & z_1:z_2:z_3=0:0:1, \\ Q_4; & z_1:z_2:z_3=1:0:0, \\ Q_5; & z_1:z_2:z_3=0:1:0. \end{array}\right\} \qquad (4)$$

Now, a transformation can be found which throws

$$\begin{array}{lcccc} & O_1 & O_2 & O_3 & O_4 \\ \text{to} & Q_2 & Q_5 & Q_3 & Q_4 \end{array}$$

in the order indicated, namely,

$$\left.\begin{array}{ll} \text{direct:} & x_1:x_2:x_3=-z_1+z_2+z_3:z_1-z_2+z_3:z_1+z_2-z_3, \\ \text{inverse:} & z_1:z_2:z_3=x_2+x_3:x_3+x_1:x_1+x_2. \end{array}\right\} \qquad (5)$$

8. Furthermore, this transformation also throws the generators

$$\begin{array}{lccc} & K' & L' & M' \\ \text{to} & K & L & M. \end{array}$$

Hence, it follows at once that K, L, M are the *generators* of $G_{24}^{(1)}$, and that Q_2, Q_3, Q_4, Q_5 are permuted among themselves by *every* transformation of the linear subgroup.

Because of the projective relation thus established between G_{41} and $G_{24}^{(1)}$, the complete quadrangle [with its diagonal lines] whose vertices are Q_2, Q_3, Q_4, Q_5 is the geometric representation for the latter group. See Fig. II.

The fundamental region† for the new configuration is derived from that defined in (2) by the transformation (5). It is defined‡ by

$$\left.\begin{array}{r} z_1-z_2\geqq 0, \\ z_1-z_3\leqq 0, \\ z_1+z_2-z_3\geqq 0. \end{array}\right\} \qquad (6)$$

The new generators are, as they should be, edge-operators with reference to this region.

---

*For convenience, Fig's. I, VII, VIII, IX are placed together on the last page of plates.

†An independent determination of the fundamental region (6) is given in Arts. 51 to 57, where the transformation names of all the regions are derived, and certain generational relations are discovered from the configuration itself.

‡Where z_1, z_2, z_3 are all + above $z_2=0$ and to the right of $z_1=0$.

The plane is divided into 24 regions which are permuted among themselves by every transformation of the group. That is, if we call this configuration Π_1,

$$G_{24}^{(1)}\Pi_1 = \Pi_1. \tag{7}$$

§3.—*The Subgroups Conjugate with $G_{24}^{(1)}$.*

9. The general theory* of Cremona transformations may be illustrated for the quadratic operators of G_{120} by considering the product $MT \sim (1534)$† [Art. 4],

$$z'_1 : z'_2 : z'_3 = (z_1 - z_2)(z_3 - z_2) : z_3(z_1 - z_2) : z_1(z_3 - z_2),$$

of which the inverse is

$$z_1 : z_2 : z_3 = z'_3(z'_2 - z'_1) : (z'_2 - z'_1)(z'_3 - z'_1) : z'_2(z'_3 - z'_1). \tag{1}$$

In the z-plane, the fundamental points are

$$\left.\begin{array}{ll} Q_2; & z_1 : z_2 : z_3 = 1 : 1 : 1, \\ Q_3; & z_1 : z_2 : z_3 = 0 : 0 : 1, \\ Q_4; & z_1 : z_2 : z_3 = 1 : 0 : 0, \end{array}\right\} \tag{2}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q_2 Q_3; & z_1 - z_2 = 0, \\ Q_2 Q_4; & z_2 - z_3 = 0, \\ Q_3 Q_4; & z_2 = 0, \end{array}\right\} \tag{3}$$

while in the z'-plane the fundamental points are

$$\left.\begin{array}{ll} Q'_2; & z'_1 : z'_2 : z'_3 = 1 : 1 : 1, \\ Q'_3; & z'_1 : z'_2 : z'_3 = 0 : 0 : 1, \\ Q'_5; & z'_1 : z'_2 : z'_3 = 0 : 1 : 0, \end{array}\right\} \tag{4}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q'_2 Q'_3; & z'_1 - z'_2 = 0, \\ Q'_2 Q'_5; & z'_1 - z'_3 = 0, \\ Q'_3 Q'_5; & z'_1 = 0, \end{array}\right\} \tag{5}$$

---

* Clebsch, "Vorlesungen über Geometrie," vol I, pp. 474–496.

† Read, "The Operator *MT* Corresponding to the Substitution (1534)."

10. Evidently any operator whose square is identity, has the *same* fundamental points and lines in *both* planes (considered as superposed), though they are not necessarily associated in the same order in the two planes.

Indeed, this property is true of any one of the set of 6 transformations [Art. 19]

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i),^{*} \quad [i \neq j, \neq k, \neq l, = 2, 3 \ldots . 5],$$

In particular, any one of the set D_{12}^{-1} has the *coordinate vertices and sides* in some order as its fundamental points and lines in each plane.

Of these the quadratic inversion [Art. 4],

$$T' \sim (12); \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2$$

is the simplest, having always a vertex associated with its *opposite* side in the coordinate triangle.

11. Consider the transformation under MT from the z-plane to the z'-plane:

(a) A straight line (in general)

$$\alpha z_1 + \beta z_2 + \gamma z_3 = 0$$

goes into a *non-degenerate* conic

$$\alpha z_3' (z_2' - z_1') + \beta (z_2' - z_1')(z_3' - z_1') + \gamma z_2' (z_3' - z_1') = 0,$$

passing through the three points (4).

(b) A line through one of the points (2)

$$\alpha z_3 + \beta (z_1 - z_2) = 0$$

goes into a *degenerate* conic consisting of another line through *some one of the points* (4) *and its opposite fundamental side,*

$$z_1' - z_3' = 0,$$
$$\alpha z_3' - (\alpha - \beta) z_1' = 0.$$

In particular, a *side* of the quadrangle Π_1 through *one* fundamental point goes into a degenerate conic consisting of *two sides*, thus:

$$z_1 - z_3 = 0$$

---

[*] I use the notation of Cayley, Quarterly Journal, vol. 25, p. 78, except that I use the parentheses to indicate substitutions, and hence, indicate groups by { }, so that the above form means the group $\{jkl\}$ all multiplied by the substitution $(1i)$ while $\{jkl\}$ all $\{1i\}$ means the same group multiplied by the group 1, $(1i)$.

The plane is divided into 24 regions which are permuted among themselves by every transformation of the group. That is, if we call this configuration Π_1,

$$G_{24}^{(1)}\Pi_1 = \Pi_1. \tag{7}$$

§3.—*The Subgroups Conjugate with $G_{24}^{(1)}$.*

9. The general theory* of Cremona transformations may be illustrated for the quadratic operators of G_{120} by considering the product $MT \sim (1534)$† [Art. 4],

$$z'_1 : z'_2 : z'_3 = (z_1 - z_2)(z_3 - z_2) : z_3(z_1 - z_2) : z_1(z_3 - z_2),$$

of which the inverse is

$$z_1 : z_2 : z_3 = z'_3(z'_2 - z'_1) : (z'_2 - z'_1)(z'_3 - z'_1) : z'_2(z'_3 - z'_1). \tag{1}$$

In the z-plane, the fundamental points are

$$\left.\begin{array}{ll} Q_2; & z_1 : z_2 : z_3 = 1:1:1, \\ Q_3; & z_1 : z_2 : z_3 = 0:0:1, \\ Q_4; & z_1 : z_2 : z_3 = 1:0:0, \end{array}\right\} \tag{2}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q_2 Q_3; & z_1 - z_2 = 0, \\ Q_2 Q_4; & z_2 - z_3 = 0, \\ Q_3 Q_4; & z_2 = 0, \end{array}\right\} \tag{3}$$

while in the z'-plane the fundamental points are

$$\left.\begin{array}{ll} Q'_2; & z'_1 : z'_2 : z'_3 = 1:1:1, \\ Q'_3; & z'_1 : z'_2 : z'_3 = 0:0:1, \\ Q'_5; & z'_1 : z'_2 : z'_3 = 0:1:0, \end{array}\right\} \tag{4}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q'_2 Q'_3; & z'_1 - z'_2 = 0, \\ Q'_2 Q'_5; & z'_1 - z'_3 = 0, \\ Q'_3 Q'_5; & z'_1 = 0, \end{array}\right\} \tag{5}$$

---

* Clebsch, "Vorlesungen über Geometrie," vol I, pp. 474–496.

† Read, "The Operator *MT* Corresponding to the Substitution (1534)."

10. Evidently any operator whose square is identity, has the *same* fundamental points and lines in *both* planes (considered as superposed), though they are not necessarily associated in the same order in the two planes.

Indeed, this property is true of any one of the set of 6 transformations [Art. 19]

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i),^{*} \quad [i \neq j, \neq k, \neq l, = 2, 3 \ldots . 5],$$

In particular, any one of the set D_{12}^{-1} has the *coordinate vertices and sides* in some order as its fundamental points and lines in each plane.

Of these the quadratic inversion [Art. 4],

$$T' \sim (12)\,; \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2$$

is the simplest, having always a vertex associated with its *opposite* side in the coordinate triangle.

11. Consider the transformation under MT from the z-plane to the z'-plane:

(a) A straight line (in general)

$$\alpha z_1 + \beta z_2 + \gamma z_3 = 0$$

goes into a *non-degenerate* conic

$$\alpha z_3' (z_2' - z_1') + \beta (z_2' - z_1')(z_3' - z_1') + \gamma z_2' (z_3' - z_1') = 0,$$

passing through the three points (4).

(b) A line through one of the points (2)

$$\alpha z_3 + \beta (z_1 - z_2) = 0$$

goes into a *degenerate* conic consisting of another line through *some one of the points* (4) *and its opposite fundamental side,*

$$z_1' - z_2' = 0,$$
$$\alpha z_3' - (\alpha - \beta) z_1' = 0.$$

In particular, a *side* of the quadrangle Π_1 through *one* fundamental point goes into a degenerate conic consisting of *two sides*, thus:

$$z_1 - z_3 = 0$$

---

[*] I use the notation of Cayley, Quarterly Journal, vol. 25, p. 73, except that I use the parentheses to indicate substitutions, and hence, indicate groups by { }, so that the above form means the group $\{jkl\}$ all multiplied by the substitution $(1i)$ while $\{jkl\}$ all $\{1i\}$ means the same group multiplied by the group 1, $(1i)$.

goes into $$z_1' = 0 \text{ and } z_2' - z_3' = 0.$$

(c) A *fundamental* line, joining *two* of the points (2),

$$z_1 - z_2 = 0$$

goes into a degenerate conic consisting of two of the fundamental lines (5), namely

$$z_1' = 0$$

and $$z_1' - z_2' = 0.$$

See also (f) below for the correspondence of the points on such a line.

(d) An *ordinary* point, defined by the intersection of two (general) lines, goes into the fourth (movable) point of intersection of the two corresponding conics, the other three intersections being the fundamental points in the z'-plane.

(e) One of the fundamental points (3) corresponds to one of the fundamental sides (5) in the following sense: to each "*direction*" in which a movable point may approach the given fundamental point in the z-plane, there corresponds a definite point situated on a certain one of the fundamental lines in the z'-plane. Any such "direction" is determined by a *specific tangent in the fundamental point* and the corresponding point in the z'-plane is given by (b) above. For example, to the direction given by the tangent

$$\alpha z_2 + \beta (z_1 - z_2) = 0 \tag{6}$$

in the fundamental point

$$z_1 : z_2 : z_3 = 0 : 0 : 1$$

there corresponds the point in the z'-plane given by the intersection of the line

$$\alpha z_3' - (\alpha - \beta) z_1' = 0$$

with the fundamental side

$$z_1' - z_2' = 0. \tag{7}$$

Thus, when the tangent (6) varies through all directions, the fundamental side (7) becomes the locus of the corresponding points.

(f) *A fundamental point must, therefore, be regarded as a "pencil of directions,"* and this makes clear how a fundamental line in the z-plane goes, by (c), into two fundamental lines in the z'-plane. For to each point on such a line, there corresponds, by (e), a specific direction in a certain pencil of the z'-plane. Among these points, however, are two fundamental points, to each of which corresponds, by the infinity of directions through it, a certain fundamental line in the z'-plane. These lines are themselves, therefore, two direction-tangents in the above pencil, which is, then, determined by their intersection.

12. Since, under the transformation MT, the pencils of directions (2) go, in some order, into the ranges of points (5), the former are called *the critical points* and the latter *the critical lines of the transformation.* All other points are non-critical under the transformation MT.

Likewise (4) and (3) are the critical points and lines of the inverse transformation $(MT)^{-1}$. Evidently $(MT)^{-1}$ throws any line of the plane into a conic passing through the critical points of MT.

13. Since, for every quadratic transformation in G_{120}, the critical points are certain three (n^2-1) of the four vertices, and the critical lines are three of the six sides of the complete quadrangle Π_1, previously found [Art. 8], the group is said to have four critical points and six critical lines.

Classification of the Quadratic Transformations, Arts. 14–16.

14. It has been seen that all linear transformations in G_{120} are associated with the substitutions which leave the index 1 fixed. The quadratic transformations may be divided into four sets of 24 each, according as the index 1 is thrown to 2, 3, 4 or 5 by the corresponding substitutions.

15. Lemma. *If* s_{1i} *is a transformation corresponding to any particular substitution throwing* 1 *to* i, *and if* S_{1i} *represents the whole set of such transformations, then*

$$S_{1i} = G_{24}^{(1)} s_{1i} \sim \{2345\} \text{ all } (1i), \qquad [i = 2, 3, 4, 5].$$

For the set S_{1i} must be such that when the corresponding substitutions are combined with the one belonging to s_{1i}^{-1}, there will result all substitutions leaving 1 fixed; that is, the substitution group corresponding to $G_{24}^{(1)}$, thus:

$$S_{1i}\, s_{1i}^{-1} = G_{24}^{(1)}.$$

Hence, multiplying on the right by s_{1i},

$$S_{1i} = G_{24}^{(1)} s_{1i} \tag{8}$$

or

$$S_{1i}^{-1} = (G_{24}^{(1)} s_{1i})^{-1} = s_{1i}^{-1}\, G_{24}^{(1)}. \tag{9}$$

16. **Theorem.** *The critical points are the same for all quadratic transformations belonging to the same set* S_{1i}, $(i = 2, 3, 4, 5)$.

For if s_{1i}^{-1}, any particular transformation of the set S_{1i}^{-1}, be applied to the complete quadrangle Π_1 [Art. 8] a definite configuration will result in which every line of Π_1 will have become a conic passing through three fixed points, namely, the critical points of s_{1i} [Art. 12]. If the new figure be called Π_i, $[i = 2, 3, 4, 5]$, then

$$s_{1i}^{-1}\Pi_1 = \Pi_i. \tag{10}$$

Any other transformation of the set S_{1i}^{-1}, when applied to Π_1, will produce the same figure Π_i, for, using equation (9),

$$S_{1i}^{-1}\Pi_1 = s_{1i}^{-1}\,G_{24}^{(1)}\Pi_1.$$

By equations (7) of Art. 8 and (10) above,

$$s_{1i}^{-1}\,G_{24}^{(1)}\Pi_1 = s_{1i}^{-1}\Pi_1 = \Pi_i.$$

Hence,
$$S_{1i}^{-1}\Pi_1 = \Pi_i. \tag{11}$$

Since each transformation of the set S_{1i}^{-1} throws the lines of Π_1 into conics intersecting in the same fixed points in Π_i, therefore, all the transformations in the set S_{1i} have the same critical points [Art. 12].

Configurations for the Subgroups $G_{24}^{(i)}$, Arts. 17–24.

17. If $G_{24}^{(1)}$ be transformed by any one of the set of transformations S_{1i}, the new subgroup is the same as that produced by any other transformer of the set, for if

$$s_{1i}^{-1}\,G_{24}^{(1)}\,s_{1i} = G_{24}^{(i)},$$

then, by use of (8) and (9),

$$S_{1i}^{-1}\,G_{24}^{(1)}\,S_{1i} = G_{24}^{(i)}. \tag{12}$$

This $G_{24}^{(i)}$ corresponds to the substitution group which leaves the index i fixed and permutes among themselves $1, j, k, l$, $[i \neq j, \neq k, \neq l, = 2, 3, 4, 5]$.

To $G_{24}^{(i)}$ belongs a configuration related to Π_1 in the following manner:

Let g and g' be any pair of corresponding transformations in $G_{24}^{(1)}$ and $G_{24}^{(i)}$ respectively, so that

$$s_{1i}^{-1}\,g\,s_{1i} = g'.$$

Suppose P to be any point of the plane and P' its conjugate under g', then, by (13),

$$s_{1i}^{-1}\, g\, s_{1i}\, P = P'. \tag{14}$$

If, now, $s_{1i}\, P = P_1$ and $gP_1 = P_2$, then must

$$s_{1i}^{-1}\, P_2 = P'.$$

Thus P_1 and P_2, a pair of conjugate points under g, are thrown by s_{1i}^{-1} to P and P' respectively, a pair of conjugate points under g'.

Since (13) holds for every g and for every s_{1i}^{-1}, therefore, any transformation of the set S_{1i}^{-1}, operating on a pair of conjugate points under $G_{24}^{(1)}$, gives a corresponding pair of conjugate points under $G_{24}^{(i)}$. Therefore, the configuration connected with $G_{24}^{(i)}$ is derived by applying to Π_1 any transformation of the set S_{1i}^{-1} and the result is by (11) Π_i, $\quad (i = 2 \ldots . 5)$.

18. THEOREM. *The configuration* Π_i *is thrown into itself by all transformations of* $G_{24}^{(i)}$.

For by (12), $\quad G_{24}^{(i)}\, \Pi_i = S_{1i}^{-1}\, G_{24}^{(1)}\, S_{1i}\, \Pi_i$

and by (11), $\quad \Pi_i = S_{1i}^{-1}\, \Pi_1$.

Hence, $\quad S_{1i}\, \Pi_i = S_{1i}\, S_{1i}^{-1}\, \Pi_1 = \Pi_1$,

and, Art. 8, $\quad G_{24}^{(1)}\, S_{1i}\, \Pi_i = G_{24}^{(1)}\, \Pi_1 \quad = \Pi_1$.

Then, by (11), $\quad S_{1i}^{-1}\, G_{24}^{(1)}\, S_{1i}\, \Pi_i = S_{1i}^{-1}\, \Pi_1 \quad = \Pi_i$.

Therefore, $\quad G_{24}^{(i)}\, \Pi_i = \Pi_i. \qquad (15)$

19. The notation for the 4 points in Π_1 has been so chosen [Art. 7] that the three critical points for the set of transformers S_{1i} are

$$Q_j,\; Q_k,\; Q_l. \qquad [i \neq j,\; \neq k,\; \neq l,\; = 2,\, 3,\, 4,\, 5].$$

These are the three points through which pass all conics in Π_i.

Thus Q_i plays a particular rôle in the passage from Π_1 to Π_i, in that by any of the six transformations in the set D_{1i}^{-1} [Art. 10], it is unmoved, and the other three critical points become ranges of points on

$$Q_i\, Q_j,\; Q_i\, Q_k,\; Q_i\, Q_l,$$

while, by any other* transformation of the system S_{1i}^{-1}, some one of the three is carried to Q_i and the other two with Q_i become ranges of points on

$$Q_i\,Q_j,\quad Q_i\,Q_k,\quad Q_i\,Q_l.$$

20. It follows that in operating upon Π_1 by any one of the set of transformations

$$S_{1i}^{-1}, \qquad (i = 2,\ 3,\ 4,\ 5),$$

the resulting configuration, Π_i, contains again the 4 *pencils* and 6 *sides* of Π_1, while the diagonal lines become proper conics in Π_i. It is convenient for this purpose to choose the following special transformations:

$$\begin{aligned}
s_{12} &\sim (12) \quad ; & z_1' : z_2' : z_3' &= z_2 z_3 : z_1 z_3 : z_1 z_2,\\
s_{13} &\sim (132) \quad ; & z_1' : z_2' : z_3' &= z_3 (z_3 - z_2) : z_3 (z_3 - z_1) : (z_3 - z_1)(z_3 - z_2),\\
s_{14} &\sim (1452) \; ; & z_1' : z_2' : z_3' &= z_1 (z_1 - z_3) : (z_1 - z_3)(z_1 - z_2) : z_1 (z_1 - z_2),\\
s_{15} &\sim (152) \quad ; & z_1' : z_2' : z_3' &= z_2 (z_2 - z_3) : (z_2 - z_3)(z_2 - z_1) : z_2 (z_2 - z_1).
\end{aligned}$$

The generators of the subgroups $G_{24}^{(i)}$ are found by transforming K, L, M through s_{12}, s_{13}, s_{14}, s_{15} respectively. The three conics of Π_i are given by operating upon the diagonal lines of Π_1 by s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1} respectively. The boundaries of the fundamental regions of Π_i are shown by operating upon those of Π_1 by s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1} respectively.

The configurations, $\Pi_2 \ldots\ldots \Pi_5$, are shown in the figures III, IV, V, VI.

§4.—THE CONFIGURATION FOR G_{120}.

Algebraic Study of the Configuration, Arts. 21–27.

21. If, now, the figures Π_i be superposed upon the complete quadrangle Π_1, the resulting configuration Π admits the following algebraic verification as to the intersections of the twelve conics with the sides and diagonals of Π_1.

22. Since [Art. 19] the three conics of Π_i each pass through

$$Q_j,\ Q_k,\ Q_l, \qquad [i \neq j,\ \neq k,\ \neq l,\ = 2,\ 3,\ 4,\ 5],$$

but none of them through the fourth vertex, therefore, through each vertex pass

$$3 \,.\, 3 = 9 \text{ conics}.$$

23. Since [Art. 20] the 6 sides and 4 pencils of Π_1 are reproduced in Π_i, while the 3 diagonals in each case become proper conics, the intersections of Π

---

* The set D_{1i}^{-1} is included in the set S_{1i}^{-1}. See Art 15.

will evidently be of three kinds only, namely, (a) *conics with the sides*, (b) *conics with the diagonals*, (c) *conics with conics*.

24. Each side, in addition to the 2 vertices lying on it, through each of which pass 9 conics, has 2 other rational points through each of which passes one conic, thus:

On the sides	lie the rational points.
$z_1 = 0$	$0 : 1 : 2$ and $0 : 2 : 1$
$z_1 - z_3 = 0$	$1 : -1 : 1$ and $2 : 1 : 2$
$z_2 = 0$	$1 : 0 : 2$ and $2 : 0 : 1$
$z_2 - z_3 = 0$	$-1 : 1 : 1$ and $1 : 2 : 2$
$z_1 - z_2 = 0$	$1 : 1 : -1$ and $2 : 2 : 1$
$z_3 = 0$	$1 : 2 : 0$ and $2 : 1 : 0$

25. Each side has also one intersection with a diagonal line. It will be found in the succeeding grouptheoretic study that these belong to the same class as the intersections with the conics.

26. Each diagonal line has 4 points, through each of which pass 4 conics, one belonging to each of the figures Π_i.

Thus, on the lines

$$z_1 + z_2 - z_3 = 0, \quad z_1 - z_2 - z_3 = 0, \quad z_1 - z_2 + z_3 = 0$$

lie respectively the systems of 4 points:

$-\lambda : 1 + \lambda : 1$	$\lambda : -\lambda' : 1$	$-\lambda : \lambda' : 1$
$-\lambda' : 1 + \lambda' : 1$	$\lambda' : -\lambda : 1$	$-\lambda' : \lambda : 1$
$1 + \lambda : -\lambda : 1$	$1 + \lambda : \lambda : 1$	$\lambda : 1 + \lambda : 1$
$1 + \lambda' : \lambda' : 1$	$1 + \lambda' : \lambda' : 1$	$\lambda' : 1 + \lambda' : 1$

wherein
$$\lambda = \frac{1 + \sqrt{5}}{2}, \quad \lambda' = \frac{1 - \sqrt{5}}{2}.$$

27. The only real intersections of the 12 conics among themselves are at the 4 vertices and at the above 12 irrational points on the 3 diagonals. But there is a set of 20 imaginary points, through each of which pass 3 conics. These will be discussed in the succeeding grouptheoretic study. The configuration Π is shown in Fig. X.

Grouptheoretic Study of the Configuration II, Arts. 28–46.

28. In order to discuss the systems of conjugate elements under the main group, a generating operator is needed which extends $G_{24}^{(1)}$ to G_{120}. *Any* transformation of G_{120} not contained in $G_{24}^{(1)}$ will serve, since the index of $G_{24}^{(1)}$ under G_{120} is a prime.

It will be convenient to choose for this purpose

$$T \sim (15)(34), \qquad \text{[see Art. 4]},$$

since K, L and T are edge-operators for the region defined by

$$\begin{aligned} z_1 - z_3 \qquad\qquad\qquad &\leqq 0, \\ z_1 + z_2 - z_3 \qquad\qquad &\geqq 0, \\ z_1\, z_2 - z_1\, z_3 + z_2\, z_3 &\leqq 0, \end{aligned} \qquad \text{[see Fig. XI]}.$$

The old generator M is now expressible in terms of K, L, T, thus:

$$M = TKLKTLTK,$$

so that K, L, T generate the group G_{120}.

29. Under G_{120} any element—point, line or conic—will, in general, go into 120 conjugates. Certain elements may, however, go into fewer conjugates. Such a *special* element E is invariant under a subgroup $G\{E\}$, whose index under G_{120} indicates the number of elements in the conjugate system.

The element E may be designated by a notation n consisting of such a combination of the indices 1 .... 5 as will characterize the corresponding substitution group $G'\{E\}$. This notation may be indicated on the group by $G^N\{E\}$.

Let S be any transformation in G_{120}. Then, if E is invariant under $G\{E\}$, $S^{-1}E$ will be invariant under $S^{-1}G\{E\}S$.

Let s be the substitution corresponding to S. Then, if n is the notation for E, ns will be the notation for $S^{-1}E$, wherein s acts as a substitution upon the indices in the cycles of the notation n. Likewise, the notation for SE is ns^{-1}.

The complete set of substitutions changing the notation of E to that of SE is given by $G'\{E\}s^{-1}$, and the transformations corresponding to the inverse of the substitutions in the set are the only ones in G_{120} which throw E to SE.

30. As an illustration of such a special element, the diagonal

$$z_1 + z_2 - z_3 = 0,$$

which is fixed by points under the transformation corresponding to the substitution (23)(45), is found to be invariant under the subgroup

$$G_8^{23.45} \sim \{2435\}_8.$$[*]

Its notation will then be 23 . 45, provided we agree

$$23.45 = 32.45 = 32.54 = 45.32 = 54.32 = 54.23, \text{ etc.};$$

that is, *the pairing is definitive, but not the sequence either of the pairs or the indices in a pair.*

The transformation $S \sim (134)$

throws $z_1 + z_2 - z_3 = 0$

to the conic $z_1 z_2 - 2z_2 z_3 + z_3^2 = 0.$

This conic is fixed by points under the transformation $\sim (15)(24)$ and is invariant under the subgroup

$$G_8^{15.24} = S^{-1} G_8^{23.45} S \sim (134)^{-1} \{2435\}_8 (134).$$

Its notation is $15.24 = [23.45](134)$.[†]

The complete set of transformations throwing the diagonal 23 . 45 to the conic 15 . 24 is given by the substitutions

$$[\{2435\}_8 (134)]^{-1}.$$

31. Since the index of $G_8^{23.45}$ under G_{120} is 15, it would at once appear that the 3 diagonals and 12 conics form a conjugate system. This is shown in the following table,[‡] in which the generators K, L and T are applied successively to the diagonal, while its notation is transformed by the *inverse* of the correspond-

---

[*] Notation of Cayley. See foot-note to Art. 10.

[†] Article 29. This means operate on the cycles of the *notation* 23 . 45 with the *substitution* (134).

[‡] The result of operating on any diagonal or conic of the configuration by any transformation of G_{120} may be read at once from this table without algebraic computation.

ing substitutions (these being the same as the *direct*, since K, L, T are all of period 2).

Diagonals and Conics.	Notation.	$K \sim (34)$.	$L \sim (23)(45)$.	$T \sim (15)(34)$.
$z_1 + z_2 - z_3 = 0$	23 . 45	24 . 35	23 . 45	13 . 24
$z_1 - z_2 - z_3 = 0$	24 . 35	23 . 45	24 . 35	14 . 23
$z_1 - z_2 + z_3 = 0$	25 . 34	25 . 34	25 . 34	12 . 34
$z_2^2 - z_1 z_3 = 0$	12 . 34	12 . 34	13 . 25	25 . 34
$z_1^2 - z_2 z_3 = 0$	12 . 35	12 . 45	13 . 24	14 . 25
$z_1 z_2 - z_3^2 = 0$	12 . 45	12 . 35	13 . 45	13 . 25
$z_2^2 - 2z_2 z_3 + z_1 z_3 = 0$	13 . 24	14 . 23	12 . 35	23 . 45
$z_1^2 - 2z_1 z_3 + z_2 z_3 = 0$	13 . 25	14 . 25	12 . 34	12 . 45
$z_1 z_2 - z_1 z_3 - z_2 z_3 = 0$	13 . 45	14 . 35	12 . 45	13 . 45
$z_2^2 - 2z_1 z_2 + z_1 z_3 = 0$	14 . 23	13 . 24	15 . 23	24 . 35
$z_1 z_2 - 2z_1 z_3 + z_3^2 = 0$	14 . 25	13 . 25	15 . 34	12 . 35
$z_1 z_2 + z_1 z_3 - z_2 z_3 = 0$	14 . 35	13 . 45	15 . 24	14 . 35
$z_1^2 - 2z_1 z_2 + z_2 z_3 = 0$	15 . 23	15 . 24	14 . 23	15 . 24
$z_1 z_2 - 2z_2 z_3 + z_3^2 = 0$	15 . 24	15 . 23	14 . 35	15 . 23
$z_1 z_2 - z_1 z_3 + z_2 z_3 = 0$	15 . 34	15 . 34	14 . 25	15 . 34

The 3 diagonals

$$2i\,.\,jk \qquad\qquad (i, j, k, = 3, 4, 5)$$

belong to the *linear* subgroups

$$G_8^{2i.jk} \sim \{2jik\}_8.$$

The 12 conics $1i.jk$ $(i, j, k, = 2, 3, 4, 5)$ belong the *quadratic* subgroups

$$G_8^{1i.jk} \sim \{1jik\}_8.$$

32. Again, the side $z_1 = 0$

is found to be *fixed by points* under the transformation $\sim (24)$ and invariant under the subgroup

$$G_{12}^{24} \sim \{135\} \text{ all } \{24\}.\text{[*]}$$

Its notation will then be $24 = 42.$

Since the index of G_{12}^{24} is 10, *the* 6 *sides do not form a complete system,* but we have seen that under quadratic transformations a *pencil of directions* plays the same rôle as a *side of the quadrangle* Π_1, so that the 4 pencils and 6 sides may form a conjugate system. In fact the transformation

$$S \sim (12)(34)$$

throws the side $z_1 = 0$

to the pencil at $z_1 : z_2 : z_3 = 0 : 0 : 1.$

This pencil is fixed by *directions* under the transformation $\sim (13)$ and is invariant under the subgroup

$$G_{12}^{13} = S^{-1} G_{12}^{24} S \sim [(12)(34)]^{-1} [\{135\} \text{ all } \{24\}] [(12)(34)].$$

The notation for the pencil is

$$13 = 24\, [(12)(34)].\text{[†]}$$

The complete set of operators throwing the side 24 into the pencil 13 is given by the substitutions

$$[\{135\} \text{ all } \{24\}] [(12)(34)].$$

33. The notation for the 10 elements and the proof that they form a closed system under the generators K, L, T, is as follows:

---

[*] See foot-note to Art. 10.

[†] This means operate upon the *notation* 24 with the *substitution* (12)(34).

Sides and Pencils.		Notation.	$K \sim (34)$.	$L \sim (23)(45)$.	$T \sim (15)(34)$.
$1:1:1$		12	12	13	25
$0:0:1$		13	14	12	45
$1:0:0$		14	13	15	35
$0:1:0$		15	15	14	15
z_3	$=0$	23	24	23	24
z_1	$=0$	24	23	35	23
z_2	$=0$	25	25	34	12
$z_1 - z_3$	$=0$	34	34	25	34
$z_2 - z_3$	$=0$	35	45	24	14
$z_1 - z_2$	$=0$	45	35	45	13

The 6 sides,

$$ij = ji, \qquad (i \neq j, = 2 \,\ldots\ldots\, 5),$$

belong to the subgroups

$$G_{12}^{ij} \sim \{1kl\} \text{ all } \{ij\}, \qquad (k, l \neq i, j, = 2 \,\ldots\ldots\, 5).$$

The 4 pencils $\quad 1i = i1 \quad (i = 2 \,\ldots\ldots\, 5),$

belong to the subgroups

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}, \qquad (j, k, l, \neq i, = 2 \,\ldots\ldots\, 5).$$

34. Each line or conic in the above system is a *locus of fixed points* under the transformation corresponding to the substitution by which it is named. These are the *only such loci* under the operators of G_{120}, as will at once appear by the following complete list of fixed points for the different types of transformations:

35. *Period* 2. *Two Types.*

(1). *Type* (ij).

(a). When $i \neq j, = 2 \ldots 5$, the side ij is a *locus of fixed points.*

One point not in the locus ij is fixed, namely, the intersection of the diagonal $ij.kl$ with the side kl. $(k, l, \neq i, j, = 2 \ldots 5)$.

One direction is fixed in each of the pencils, $1k$ and $1l$, namely, that one whose tangent passes through the fixed point outside the locus ij.

(b). When $i = 2 \ldots 5$, the pencil $1i$ is a *centre* of *fixed directions* under the transformations $s_{1i} \sim (1i)$.

One direction is also fixed under these transformations in each of the pencils

$$1j, \ 1k, \ 1l, \qquad (j, k, l, \neq i, = 2 \ldots 5).$$

And one point is fixed in each of the sides passing through the vertex $1i$, namely,

$$jk, \ jl, \ kl.$$

(2). *Type* $(ij)(kl)$.

(a). When $i, j, k, l, = 2 \ldots 5$, the diagonal $ij.kl$ is a *locus* of *fixed points* under the transformation $\sim (ij)(kl)$ and one point not in $ij.kl$ is fixed under the same transformation, namely, the intersection of the other two diagonals.

(b). When $i = 1$ and $j, k, l = 2 \ldots 5$, the conic $1j.kl$ is a *locus* of *fixed points* under the transformation $\sim (1j)(kl)$, and one direction also is fixed in the pencil $1i$, that one whose tangent is common to the conics

$$1k.jl \text{ and } 1l.jk.$$

36. *Period* 3. *One type,* (ijk). $(i, j, k = 1 \ldots 5)$.

There is no locus of fixed points for this type, but there are two classes of *discrete fixed* points, namely,

(a). A pair of imaginary points through which pass the three conics [Art. 42]

$$ij.lm, \ ik.lm, \ jk.lm, \qquad (i, j, k, l, m, = 1 \ldots 5).$$

(b). A pair of imaginary points lying on the side (or imaginary directions in the pencil) (Art. 46)

$$lm, \qquad (l, m = 1 \ldots 5).$$

37. *Period* 4. *One type*, $(ijkl)$.

There is no locus of fixed points, but again two classes of *discrete* fixed points.

(a). Real points where two sides meet two diagonals. Such a *point* on a *side* is conjugate with a *direction* in a *pencil* whose tangent (one of the sides) is common to the two conics corresponding to the two diagonals. See Art. 43.

(b). *Imaginary points* which lie in pairs on the diagonals or conics. See Art. 45.

38. *Period* 5. *One type*, $(ijklm)$.

The only fixed points are real, discrete points whose coördinates involve the surd $\sqrt{5}$. These will be discussed in Art. 44.

39. *Period* 6. *One type*, $(ij)(klm)$.

Again, the only fixed points are imaginary, discrete points of the same form as class (b) under Period 3. See Art. 46.

40. We have thus found no loci of fixed points (or directions) except the 10 sides (and pencils) and 15 diagonals and conics which are exactly all the curves in the configuration Π.

In order now to discover all the systems of conjugate points, with their respective notations, we take any known fixed point under a given type of transformation and find the subgroup which leaves it unmoved and determine its notation by means of the corresponding substitution group. Then we operate upon it with the generators of G_{120}, transforming the notation as in Arts. 31, 33, and continue the process till the system is closed.

The intersection points of Π are classified as double, triple, quadruple and quintuple, according to the number of lines or conics passing through them.

Corresponding to a *double point lying on one of the sides will be a direction in one of the pencils, whose tangent belongs to one conic only, while to a quadruple point on one of the sides corresponds a direction in a pencil, whose tangent is common to two conics and coincides with a side passing through the pencil.*

Following is a complete enumeration of intersection points and their conjugate systems:

41. *Double Points.*

The point $$-1:0:1$$ is invariant under transformations of Period 2 corresponding to
$$(34),\ (25) \text{ and } (25)(34),$$
and under no others except identity. It, therefore, belongs to the subgroup
$$G_4^{25.34} \sim \{(25)(34)\}.$$

The point $$1:2:1$$ is also invariant under the same subgroup. The notations for these two points are, then,
$$\overline{25}.34 \text{ and } \overline{34}.25,$$
where the *sequence of pairs is definitive but not the sequence of indices in a pair.* The first pair indicates the name of the side which is cut at the point by the diagonal 25.34. The stroke distinguishes the notation from that of the diagonal $25.34 = 34.25$. [Art. 30.]

The complete list, consisting of 12 directions in the 4 pencils plus 18 points on the 6 sides is as follows:

"Direction" Tangents in the Pencils.		Notation.
$z_1 + z_3 = 0$	15	$\overline{12}.34$
$2z_1 - z_3 = 0$	15	$\overline{14}.23$
$z_1 - 2z_3 = 0$	15	$\overline{13}.24$
$z_2 + z_3 = 0$	14	$\overline{12}.35$
$2z_2 - z_3 = 0$	14	$\overline{15}.23$
$z_2 - 2z_3 = 0$	14	$\overline{13}.25$
$z_1 + z_2 = 0$	13	$\overline{12}.45$
$z_1 - 2z_2 = 0$	13	$\overline{15}.24$
$2z_1 - z_2 = 0$	13	$\overline{14}.25$
$z_1 + z_2 - 2z_3 = 0$	12	$\overline{13}.45$
$z_1 - 2z_2 + z_3 = 0$	12	$\overline{15}.34$
$-2z_1 + z_2 + z_3 = 0$	12	$\overline{14}.35$

Intersections on the	Sides.	Notation.
1 : 2 : 0	23	$\overline{23}.14$
2 : 1 : 0	23	$\overline{23}.15$
1 : —1 : 0	23	$\overline{23}.45$
0 : 2 : 1	24	$\overline{24}.13$
0 : 1 : 2	24	$\overline{24}.15$
0 : —1 : 1	24	$\overline{24}.35$
2 : 0 : 1	25	$\overline{25}.13$
1 : 0 : 2	25	$\overline{25}.14$
—1 : 0 : 1	25	$\overline{25}.34$
1 : —1 : 1	34	$\overline{34}.12$
2 : 1 : 2	34	$\overline{34}.15$
1 : 2 : 1	34	$\overline{34}.25$
—1 : 1 : 1	35	$\overline{35}.12$
1 : 2 : 2	35	$\overline{35}.14$
2 : 1 : 1	35	$\overline{35}.24$
1 : 1 : —1	45	$\overline{45}.12$
2 : 2 : 1	45	$\overline{45}.13$
1 : 1 : 2	45	$\overline{45}.23$

These points or directions, which lie by threes on the 6 sides and 4 pencils, are fixed by pairs under the 15 *distinct* conjugate subgroups of order 4. That is, the points

$$\overline{ij}.kl \text{ and } \overline{kl}.ij$$

belong to the groups $G_4^{ij.kl} \sim \{(ij)(kl)\}$. $[i, j, k, l, = 1 \ldots . 5]$.

42. *Triple points.*

The two imaginary points*

$$1-\omega : 1-\omega^2 : 1 \text{ and } 1-\omega^2 : 1-\omega : 1$$

are fixed under the transformation $\sim (254)$, and are invariant under the subgroup

---

* ω and ω^2 are imaginary cube roots of unity.

$$G_6^{13.254} \sim [\{254\} \text{ all } \{12\}] \text{ pos.}$$

These points may be named respectively

$$13 . 245 \text{ and } 13 . 254,$$

where $13 . 245 = 31 . 254 \neq 31 . 245$

and $13 . 254 = 31 . 245 \neq 31 . 254.$

That is, the meaning of the notation is unchanged by reversing the cyclic order of indices in *both parts simultaneously*, but *not in either part alone.*

The points of this system are as follows:

20 Triple Points.	Notation.
$\omega : \omega^2 : 1$	12 . 345
$\omega^2 : \omega : 1$	12 . 354
$1 - \omega : 1 - \omega^2 : 1$	13 . 245
$1 - \omega^2 : 1 - \omega : 1$	13 . 254
$1 - \omega^2 : -3\omega^2 : 3$	14 . 235
$1 - \omega : -3\omega : 3$	14 . 253
$-3\omega^2 : 1 - \omega^2 : 3$	15 . 234
$-3\omega : 1 - \omega : 3$	15 . 243
$1 - \omega^2 : 1 - \omega : 3$	23 . 145
$1 - \omega : 1 - \omega^2 : 3$	23 . 154
$1 - \omega : -\omega : 1$	24 . 135
$1 - \omega^2 : -\omega^2 : 1$	24 . 153
$-\omega : 1 - \omega : 1$	25 . 134
$-\omega^2 : 1 - \omega^2 : 1$	25 . 143
$\omega : -\omega^2 : 1$	34 . 125
$\omega^2 : -\omega : 1$	34 . 152
$-\omega^2 : \omega : 1$	35 . 124
$-\omega : \omega^2 : 1$	35 . 142
$-\omega : -\omega^2 : 1$	45 . 123
$-\omega^2 : -\omega : 1$	45 . 132

These points belong in pairs to the 10 distinct conjugate subgroups

$$G_6^{ij.klm} = [\{klm\} \text{ all } \{ij\}] \text{ pos.}$$

Namely, to $G_6^{ij.klm}$ belong the 2 points

$$ij.klm \text{ and } ij.kml, \quad [i, j, k, l, m, = 1 \ldots . 5].$$

43. *Quadruple points.*

The point $0:1:1$

is fixed under the transformations corresponding to

$$(24)(35) \text{ and } (2345),$$

and is invariant under the subgroup

$$G_8^{24.35} \sim \{2345\}_8.$$

And the direction in the pencil 15,

$$z_1 - z_3 = 0,$$

is fixed under the transformations

$$(15)(34) \text{ and } (1354),$$

and is invariant under the subgroup

$$G_8^{15.34} \sim \{1354\}_8.$$

We name this point and direction

$$\overline{24}.\overline{35} \text{ and } \overline{15}.\overline{34}$$

respectively, where the *pairing only* is definitive, the strokes serving to distinguish these from the conics 15 . 34, 24 . 35, and also from the double points

$$\overline{15}.34, \quad \overline{34}.15, \quad \overline{24}.35, \quad \overline{35}.24.$$

The complete set is as follows:

"Direction" Tangents in the Pencils.		Notation.
$z_1 - z_3 = 0$	12	$\overline{12}\,.\,\overline{34}$
$z_2 - z_3 = 0$	12	$\overline{12}\,.\,\overline{35}$
$z_1 - z_2 = 0$	12	$\overline{12}\,.\,\overline{45}$
$z_1 = 0$	13	$\overline{13}\,.\,\overline{24}$
$z_2 = 0$	13	$\overline{13}\,.\,\overline{25}$
$z_1 - z_2 = 0$	13	$\overline{13}\,.\,\overline{45}$
$z_3 = 0$	14	$\overline{14}\,.\,\overline{23}$
$z_2 = 0$	14	$\overline{14}\,.\,\overline{25}$
$z_2 - z_3 = 0$	14	$\overline{14}\,.\,\overline{35}$
$z_3 = 0$	15	$\overline{15}\,.\,\overline{23}$
$z_1 = 0$	15	$\overline{15}\,.\,\overline{24}$
$z_1 - z_3 = 0$	15	$\overline{15}\,.\,\overline{34}$
$0:1:1$		$\overline{24}\,.\,\overline{35}$
$1:0:1$		$\overline{25}\,.\,\overline{34}$
$1:1:0$		$\overline{23}\,.\,\overline{45}$

The points or directions of this system, $\overline{ij}\,.\,\overline{kl}$, are invariant under the 15 distinct subgroups

$$G_8^{ij.kl} \sim \{ikjl\}_8, \qquad (i, j, k, l = 1 \ldots . 5).$$

44. *Quintuple points.*

The points* $\quad -\lambda : 1 + \lambda : 1$ and $-\lambda' : 1 + \lambda' : 1$

are fixed under the transformation of period 5 corresponding to

$$(12543),$$

---

* $\lambda = \frac{1+\sqrt{5}}{2}, \quad \lambda' = \frac{1-\sqrt{5}}{2}$.

and are invariant under the subgroup

$$G_{10}^{12543} \sim \{12543\}_{10} \equiv [\{12543\}_{20}] \text{ pos.}$$

These points may be named respectively

12543 and 14235,

with the understanding that the meaning of the notation is unchanged so long as the same *direct* or *reverse cyclic order* is maintained, thus

$$12543 = 25431 = 54321 = \dots\dots = 13452 = 34521 = \dots\dots$$
$$14235 = 42351 = 23514 = \dots\dots = 15324 = 53241 = \dots\dots$$

This system is as follows:

Quintuple Points.	Notation.
$-\lambda : 1+\lambda : 1$	12543
$-\lambda' : 1+\lambda' : 1$	14235
$1+\lambda : \quad -\lambda : 1$	12453
$1+\lambda' : \quad -\lambda' : 1$	14325
$\lambda : \quad -\lambda' : 1$	15423
$\lambda' : \quad -\lambda : 1$	12534
$1+\lambda : \quad \lambda : 1$	15243
$1+\lambda' : \quad \lambda' : 1$	14532
$-\lambda : \quad \lambda' : 1$	15342
$-\lambda' : \quad \lambda : 1$	14523
$\lambda : 1+\lambda : 1$	13524
$\lambda' : 1+\lambda' : 1$	12345

These points are fixed by twos under the 6 distinct conjugate subgroups; thus the two points

$1ijkl$ and $1kilj$

are fixed under the subgroup

$$G_{10}^{1ijkl} \sim [\{1ijkl\}_{20}] \text{ pos.}, \qquad (i, j, k, l, = 2 \dots\dots 5),$$

45. There remain for consideration two systems consisting of imaginary fixed points which are not intersection points in the configuration II.

One such system belongs to transformations of Period 4. Thus, the points*

$$1-i:-i:1 \text{ and } 1+i:i:1$$

are fixed under the transformation $\sim(2345)$, and are invariant under the subgroup

$$G_4^{2345} \sim \{2345\} \text{ cyc.}$$

These points may be named respectively

$$2345 \text{ and } 2543,$$

with the understanding that the meaning of the notation is unchanged so long as the same direct cyclic order is maintained, thus:

$$2345 = 3452 = 4523 = 5234,$$
$$2543 = 5432 = 4325 = 3254.$$

This system is as follows:

80 Complex Points.	Notation.
$2:1+i:1$	1234
$2:1-i:1$	1432
$1:1+i:2$	1243
$1:1-i:2$	1342
$-1:-i:1$	1324
$-1:+i:1$	1423
$1+i:2:1$	1235
$1-i:2:1$	1532
$1+i:1:2$	1253
$1-i:1:2$	1352
$-i:-1:1$	1325
$+i:-1:1$	1523
$1:2:1+i$	1245
$1:2:1-i$	1542

---

* $i=\sqrt{-1}$.

80 Complex Points.	Notation.
$2 : 1 : 1 + i$	1254
$2 : 1 : 1 - i$	1452
$+ i : - i : 1$	1425
$- i : + i : 1$	1524
$1 : 1 + i : 1 - i$	1345
$1 : 1 - i : 1 + i$	1543
$1 + i : 1 : 1 - i$	1354
$1 - i : 1 : 1 + i$	1453
$1 - i : 1 + i : 1$	1435
$1 + i : 1 - i : 1$	1534
$1 - i : - i : 1$	2345
$1 + i : + i : 1$	2543
$- i : 1 - i : 1$	2354
$+ i : 1 + i : 1$	2453
$1 : - i : 1 - i$	2435
$1 : + i : 1 + i$	2534

These points lie in pairs on the 15 conics and diagonals, namely, the two points

$$jklm \text{ and } jmlk$$

lie on

$$jl \, . \, km.$$

They also belong in pairs to the 15 distinct conjugate subgroups

$$G_4^{jklm} \sim \{jklm\} \text{ cyc.,} \qquad (j, k, l, m, = 1 \ldots . 5).$$

46. Finally, the points

$$-\omega : 0 : 1 \text{ and } -\omega^2 : 0 : 1$$

are fixed under transformation of Periods 3 and 6, and belong to the subgroup

$$G_6^{134} \sim \{134\} \text{ cyc. } \{25\}.$$

And the directions

$$\omega z_1 + z_2 = 0 \text{ and } \omega^2 z_1 + z_2 = 0$$

are invariant under

$$G_6^{245} \sim \{245\} \text{ cyc. } \{13\}.$$

These points and directions may be named respectively

$$134,\ 143,\ 245,\ 254,$$

the *direct cyclic order only being definitive.*

This system is as follows:

"Direction" Tangents in the	Pencils.	Notation.
$\omega z_1 + z_2 + \omega^2 z_3 = 0$	12	345
$\omega^2 z_1 + z_2 + \omega z_3 = 0$	12	354
$\omega z_1 + z_2 = 0$	13	245
$\omega^2 z_1 + z_2 = 0$	13	254
$z_2 + \omega z_3 = 0$	14	235
$z_2 + \omega^2 z_3 = 0$	14	253
$z_1 + \omega z_3 = 0$	15	234
$z_1 + \omega^2 z_3 = 0$	15	243
Complex Points on the	Sides.	
$1 : -\omega^2 : 0$	23	145
$1 : -\omega : 0$	23	154
$0 : -\omega^2 : 1$	24	135
$0 : -\omega : 1$	24	153
$-\omega : 0 : 1$	25	134
$-\omega^2 : 0 : 1$	25	143
$1 : -\omega^2 : 1$	34	125
$1 : -\omega : 1$	34	152
$-\omega : 1 : 1$	35	124
$-\omega^2 : 1 : 1$	35	142
$-\omega : -\omega : 1$	45	123
$-\omega^2 : -\omega^2 : 1$	45	132

These points or directions lie in pairs on the 10 sides and pencils, and belong in pairs to the 10 distinct conjugate subgroups; thus the points *ijk* and *ikj* belong to

$$G_6^{ijk} = G_6^{ikj} \sim \{ijk\} \text{ cyc. } \{lm\},$$
$$(i, j, k, l, m, = 1 \ldots . 5).$$

Geometric Study of the Configuration II, Arts. 47–50.

47. In the preceding grouptheoretic study, we have found loci of two types:

(a) The 6 + 4 sides and pencils ij.
(b) The 3 + 12 diagonals and conics $ij\,.\,kl$;

and real intersection points of three types:

(a) The 30 double points $\overline{ij}\,.\,kl$.
(b) The 15 quadruple points $\overline{ij}\,.\,\overline{kl}$.
(c) The 12 quintuple points $ijklm$, $[i, j, k, l, m, = 1 \ldots . 5]$.

48. These points are distributed in the following manner:

(1) On the line of type $ij\,.\,kl$ are two points,

$$\overline{ij}\,.\,kl \text{ and } \overline{kl}\,.\,ij.$$

On the line (or pencil) of type ij are 3 points or directions,

$$\overline{ij}\,.\,kl,\quad \overline{ij}\,.\,km,\quad \overline{ij}\,.\,lm.$$

Hence we enumerate

$$2\,.\,15 = 3\,.\,10 = 30 \text{ double points.}$$

(2) On the line $ij\,.\,kl$ are 2 points

$$\overline{ik}\,.\,\overline{jl} \text{ and } \overline{il}\,.\,\overline{jk},$$

and through such a point $\overline{ik}\,.\,\overline{jl}$ pass 2 curves

$$ij\,.\,kl \text{ and } il\,.\,kj.$$

On the line of type ij are 3 points,

$$\overline{ij}\,.\,\overline{kl},\quad \overline{ij}\,.\,\overline{km},\quad \overline{ij}\,.\,\overline{lm},$$

and through such a point $\overline{ij}\,.\,\overline{kl}$ pass 2 lines

$$ij \text{ and } kl,$$

thus giving

$$\frac{2\,.\,15}{2} = \frac{3\,.\,10}{2} = 15 \text{ quadruple points.}$$

(3) On a curve of type $ij\,.\,kl$ are 4 points,

$$miklj,\quad milkj,\quad mkijl,\quad mkjil,$$

and through such a point *mikl**j* pass 5 curves,

$$ij.kl,\quad mi.kj,\quad mk.lj,\quad ml.ik,\quad mj.il,$$

thus enumerating

$$\frac{4\cdot 15}{5} = 12 \text{ quintuple points.}$$

49. A line of type *ij* contains 6 intersection points, always in the cyclic order,

$$4,\ 2,\ 4,\ 2,\ 4,\ 2, \qquad\qquad [\text{see Fig. X}],$$

where the numbers indicate the multiplicity of the points, thus:

(a) On the finite side 25:

$$\overline{14}.\overline{25},\ \overline{25}.34,\ \overline{13}.\overline{25},\ \overline{25}.14,\ \overline{25}.\overline{34},\ \overline{25}.13.$$

(b) On the line at infinity 23, starting at $\overline{23}.45$, the intersection with the diagonal 23.45, and reading clockwise,

$$\overline{23}.45,\ \overline{15}.\overline{23},\ \overline{23}.14,\ \overline{23}.\overline{45},\ \overline{23}.15,\ \overline{14}.\overline{23}.$$

(c) In a finite pencil 13, starting at $\overline{13}.\overline{25}$, the intersection with the side 24, and reading counter-clockwise [see Fig. XI],

$$\overline{13}.\overline{25},\ \overline{13}.24,\ \overline{13}.\overline{45},\ \overline{13}.25.\overline{13}.\overline{24},\ \overline{13}.45,$$

(d) In a pencil at infinity 14, starting at $\overline{14}.\overline{23}$, where the parabolas 12.34 and 13.24 are tangent to the line at infinity 23, and reading counter-clockwise,

$$\overline{14}.\overline{23},\ \overline{14}.25,\ \overline{14}.\overline{35},\ \overline{14}.23,\ \overline{14}.\overline{25},\ \overline{14}.35.$$

50. A line of type *ij*.*kl* has 8 intersection points always in the cyclic order,

$$2,\ 5,\ 4,\ 5,\ 2,\ 5,\ 4,\ \overline{5},$$

the numbers indicating the multiplicity, thus:

(a) On a diagonal 23.45:

$$\overline{23}.45,\ 12543,\ \overline{24}.\overline{35},\ 14325,\ \overline{45}.23,\ 14235,\ \overline{25}.\overline{34},\ 12453.$$

(b) On the hyperbola 15.23, starting in the pencil 15 and reading continuously along the curve

$$\overline{15}.23,\ 14523,\ \overline{12}.\overline{35},\ 15243,\ \overline{23}.15,\ 15342,\ \overline{13}.\overline{25},\ 14532.$$

(c) On an equilateral hyperbola 15 . 34, starting in the pencil 14,

$\overline{14}.\overline{35}$, 12543, $\overline{15}.34$, 12534, $\overline{13}.45$, 14235, $\overline{34}.15$, 15423.

(d) On a parabola 12 . 35, starting at infinity and reading counter-clockwise,

$\overline{15}.\overline{23}$, 12543, $\overline{35}.12$, 12345, $\overline{13}.\overline{25}$, 14235, $\overline{12}.35$, 13524.

It is to be noted here that $\overline{15}.\overline{23}$ is a direction in the pencil 15, whose tangent 23 is common to the two parabolas,

12 . 35 and 13 . 25,

and thus the pencil 15 is said to be cut by

23, 12 . 35 and 13 . 25,

forming the equivalent of a quadruple point. [Art. 40.]

Fundamental Region for G_{120}. Arts. 51–65.

51. If the fundamental region is known for any subgroup, that of the main group may be found by successively extending the subgroup and subdividing its region.

Let F_i represent the fundamental region for the subgroup G_i, then for

$$G_1 = \{1\}, \quad F_1 = \text{the whole plane.}$$

52. The extender

$$L \sim (23)(45)$$

leads to the subgroup

$$G_2 = \{L\} \sim \{23 . 45\},$$

with the relation

$$L_2 = 1 .$$

F_2 is then defined* by

$$z_3 \geqq 0,$$
$$z_1 + z_2 - z_3 \geqq 0. \qquad \text{[See Fig. VII.]}$$

53. Again, the extender

$$M \sim (45)$$

leads to the subgroup

$$G_4 = \{L, M\} \sim \{(23)(45)\}$$

---

* With the understanding that z_1, z_2, z_3 are all + above $z_2 = 0$ and to the right of $z_1 = 0$.

with the relations

$$L^3 = M^2 (LM)^3 = 1.$$

The defining inequalities of F_4 are now

$$z_3 \geqq 0,$$
$$z_1 - z_2 \geqq 0,$$
$$z_1 + z_2 - z_3 \geqq 0. \qquad \text{[See Fig. VIII.]}$$

The generators L and M are edge-operators on the boundaries of F_4, and since the repetition of these operators can introduce no new lines, F_4 is at once a *simple region.*

54. The extender

$$K \sim (34)$$

leads to the subgroup

$$G_{24}^{(1)} = \{K,\ L,\ M\} \sim \{2345\} \text{ all}$$

with the relations [see Fig. II],

$$K^3 = L^2 = M^3 = (LM)^2 = (MK)^3 = LK)^4 = 1.$$

Since the index of G_4 under $G_{24}^{(1)}$ is 6, it follows that F_4 should be partitioned into 6 simple regions by the generators of $G_{24}^{(1)}$. In fact, K, L, M are edge-operators on the lines

$$34,\ 23.45 \text{ and } 45.$$

It is easy to show that the repetition of these transformations gives no lines entering the region F_4 except

$$23,\ 34,\ 35 \text{ and } 23.45,$$

and these produce the six subdivisions defined as follows:

$$\begin{array}{rrrr}
\text{(I).} & z_1 + z_2 - z_3 \geqq 0, & z_1 - z_2 \geqq 0; & z_1 - z_3 \leqq 0. \\
\text{(II).} & z_1 - z_3 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \leqq 0. \\
\text{(III).} & z_1 - z_2 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \leqq 0. \\
\text{(IV).} & z_3 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \geqq 0. \\
\text{(V).} & z_2 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \geqq 0. \\
\text{(VI).} & z_3 \geqq 0, & z_2 \leqq 0, & z_1 + z_2 - z_3 \geqq 0.
\end{array}$$

56. By use of these inequalities and the three generators K, L, M, it may be readily shown:

(a) Each of these six regions is simple; that is, it is not subdivided by any line of Π_1.

(b) Any point in the plane *outside* one of these regions, say (1), has *one* and *only one* conjugate point within that region.

(c) Any point *within* one of these regions has no other conjugate point within it.

We may then name region (1) F_{24}, in agreement with Art. 8, (6), and the others in order will be

K, KM, KLM, KL, KLK. See Fig. IX.

Since each boundary of F_{24} is a fixed axis of reflection for one of the generators, it follows that the boundaries, including the corner points, count as a part of the region F_{24}, and hence of every conjugate region in Π_1.

57. Applying the six extenders to G_4, we get the rectangular table for $G_{24}^{(1)}$:

1	L	M	LM
K	LK	LM	LMK
KM	LKM	MKM	$LMKM$
KL	LKL	MKL	$LMKL$
KLK	$LKLK$	$MKLK$	$LMKLK$
KLM	$LKLM$	$MKLM$	$LMKLM$

These are the operators of $G_{24}^{(1)}$ as shown in Fig. II.

58. Finally, the extender

$$T \sim (15)(34)$$

leads to the main group

$$G_{120} \sim \{12345\} \text{ all.}$$

It is to be shown that F_{24} is divided into *five simple regions* by the lines and conics of the configuration Π, Fig. X.

To show what curves enter F_{24}.

(a) No curve of Π_1 can cut F_{24}, since this is a fundamental region for $G_{24}^{(1)}$.

(b) Hence, the only possibilities of partition within F_{24} are by the 12 conics.

Of these, it has been shown [§4] that none pass through the vertex $\overline{25}\,.\,\overline{34}$, none through the vertex $\overline{45}\,.\,23$, 9 through the vertex 12, one through point $\overline{34}\,.\,15$, on the boundary 34, none through any point on the boundary 45, and 4 through the point 14235 on the boundary 23 . 45.

There are thus only three possible points of entrance into F_{24},

$$12,\ \overline{34}\,.\,15 \text{ and } 14235\,.$$

It remains to show which of the conics through these points actually cut the boundary and enter the region. For this purpose it is convenient to use non-homogeneous coordinates, putting

$$\frac{z_1}{z_3}=\rho,\quad \frac{z_2}{z_3}=\sigma.$$

59. The interior of F_{24} is then defined by

$$\rho-1<0, \tag{1}$$

$$\rho-\sigma>0, \tag{2}$$

$$\rho+\sigma-1>0. \tag{3}$$

From (1) and (3), $$\sigma>0. \tag{4}$$

From (2) and (3), $$\rho>\tfrac{1}{2}. \tag{5}$$

Consider any one of the conics through 14235, say 13 . 24,

$$\sigma^2-2\sigma+\rho=0, \tag{6}$$

from which $$\rho=2\sigma-\sigma^2. \tag{7}$$

Substitute (7) in (1), $$2\sigma-\sigma^2-1<0.$$

Hence, $$(\sigma-1)^2>0. \tag{8}$$

Substitute (7) in (2). $$2\sigma-\sigma^2-\sigma>0,$$

or $$\sigma-\sigma^2>0.$$

Hence, $$\sigma>0. \tag{9}$$

which agrees with (4).

Substitute (7) in (3), $$2\sigma-\sigma^2+\sigma-1>0,$$

or $$\sigma-(\sigma-1)^2>0.$$

Hence, $$\sigma>0. \tag{10}$$

which agrees with (4).

Since, by (8), (9) and (10), the defining inequalities of F_{24} are consistent with (7), therefore, the conic 13.24 enters the region F_{24}.

In like manner the other conics

$$12.35,\quad 14.25 \text{ and } 15.24$$

may be shown to *cut* the boundary at 14235 and enter the region F_{24}.

60. Of these four conics, 15.34 passes through the point $\overline{34}.15$, and hence must *cut* the boundary 34 at that point. The other three pass through the vertex 12, and hence must pass out of F_{24} at that point.

Moreover, *none of the remaining* 9 *conics through* 12 *can enter the region* F_{24}, since the only points of exit, $\overline{34}.15$ and 14235, have already their maximum number.

Hence, precisely 4 conics pass through the region F_{24}, and, therefore, *the partition into at least* 5 *parts is established.*

61. *To show that each of these divisions is a simple region,* we name and define them in order as follows, starting at the vertex $\overline{25}.\overline{34}$ and passing along the boundary 23.45 toward the vertex $\overline{45}.23$.

$$\begin{array}{ll}
1 & ;\ \rho - 1 \leqq 0,\quad \rho + \sigma - 1 \geqq 0,\quad \rho\sigma - \rho + 1 \leqq 0. \\
T & ;\ \rho - 1 \leqq 0,\quad \rho\sigma - \rho + \sigma \geqq 0,\quad \sigma^2 - 2\sigma + \rho \geqq 0. \\
TL & ;\ \rho^2 - \sigma \geqq 0,\quad \sigma^2 - 2\sigma + \rho \leqq 0. \\
TLT & ;\ \rho^2 - \sigma \leqq 0,\quad \rho\sigma - 2\rho + 1 \geqq 0. \\
TLTL & ;\ \rho - \sigma \geqq 0;\quad \rho + \sigma - 1 \geqq 0,\quad \rho\sigma - 2\rho + 1 \geqq 0.
\end{array}$$

The only possibility of subdivision in any one of these regions is by one of the 4 conics (just shown to enter F_{24}), which is *not* a *boundary* of the region in question. For instance, the region 1, whose interior is defined by

$$\rho - 1 < 0, \tag{1}$$

$$\rho + \sigma - 1 > 0, \tag{2}$$

$$\rho\sigma - \rho + \sigma > 0, \tag{3}$$

could be subdivided only by

$$12.35;\quad \rho^2 - \sigma = 0, \tag{4}$$

or

$$13.24;\quad \sigma^2 - 2\sigma + \rho = 0, \tag{5}$$

or $$14.25;\quad \rho\sigma - 2\rho + 1 = 0. \tag{6}$$

Substitute the value of σ from (4) in (2),

$$\rho^2 + \rho - 1 > 0. \tag{7}$$

Substitute the same in (3) and find

$$\rho^2 + \rho - 1 < 0. \tag{8}$$

As (7) and (8) are contradictory, 12.35 does not enter region 1.

Substitute the value of ρ from (5) in (3),

$$\sigma^2 - 3\sigma - 1 > 0. \tag{9}$$

Also in (2), $$-\sigma^2 + 3\sigma - 1 > 0. \tag{10}$$

Add (9) and (10), $$-2 > 0.$$

As this is impossible, 12.34 has no points with region 1.

Finally, substitute the value of ρ from (6) in (3) and find

$$\sigma^2 - 3\sigma + 1 > 0. \tag{11}$$

Also in (2), $$-\sigma^2 + 3\sigma - 2 > 0. \tag{12}$$

Add (11) and (12), $$-1 > 0.$$

Hence, 14.25 does not enter the region 1. Therefore, 1 is a simple region, and in the same manner each of the other four regions may be proved simple.

The names already given to these four regions correspond to the transformations by which they are respectively derived from the region 1.

6.2. These transformations

$$1,\ T,\ TL,\ TLT,\ TLTL,$$

are the proper extenders with which to form the rectangular table for G_{120} from the operators of $G_{24}^{(1)}$, for

(a) They are all different from those of $G_{24}^{(1)}$, since no two points in F_{24} can be conjugate under $G_{24}^{(1)}$.

(b) They are distinct from one another, since the supposition of equality between any two of them leads to a contradiction.

Such a table would contain initially the four generators

$$K,\ L,\ M,\ T,$$

from which, however, M may be eliminated by the relation

$$M = TKLKTLTK, \tag{1}$$

where T, L and K are *edge-operators on the boundaries of region* 1; *that is, each generator has for its axis of reflection one of the boundary curves which it leaves fixed by points.*

The table is not given in full, but the thirty subdivisions of F_4 are marked in Fig. XI, and the others (some of which are indicated), may be read at once by applying to these the transformations of

$$G_4 = \{L, M\} \sim \{(23)(45)\},$$

together with the relation (1).

63. Since F_{24} is shown to be partitioned into 5 simple regions, therefore, so is each of the 24 divisions of Π_1.

Hence the configuration Π contains precisely 120 regions and the transformations of G_{120} throw region 1 to these 120 regions respectively, each bearing uniquely the name of the transformation by which it is derived from region 1.

Therefore, the conditions are fulfilled for a fundamental region

$$1 = F_{120},$$

in which the boundaries and vertices count as a part of the region.

(a) No two points of F_{120} are conjugate under any transformation of G_{120}, since every such operator throws each point of F_{120} to a *conjugate point in the new region whose name is the transformation in question.*

(b) Any point in the plane has *at least one* conjugate point in F_{120}; for every point belongs to one of the 120 regions, and hence is thrown to a point in F_{120} by the inverse of the transformation naming that region.

(c) No point in the plane can be conjugate to *two points* of F_{120}, since these two points would then be conjugate to each other, contrary to (a).

64. F_{120} is a triangle, two of whose sides are of type $ij.kl$, and the third of type ij, while the vertices are double, quadruple, and quintuple points respectively, thus including all types of real points and lines in the configuration Π.

Hence, each of the 120 regions possesses the same characteristics. The *apparently* two-sided figures have a *pencil* as the third side of type $1i$, and *directions* in that pencil as the *double* and *quadruple* vertices. [See Art. 40.]

Thus, region TL has the boundaries

$$13.24,\ 12.35 \text{ and the pencil } 12$$

and it has the vertices

$$14235,\ \overline{12}.\overline{34} \text{ and } \overline{12}.35.$$

65. Certain generational relations may now be read from the combination of transformations about the edges and vertices of the fundamental region.

(1) Since K, L and T are reflections on the three boundaries of F_{120}, then

$$K^2 = L^2 = T^2 = 1.$$

(2) KT is a rotation about a double point in either direction through 180°.

(3) KL is a clockwise rotation about a quadruple point through 90°.

(4) LT is a clockwise rotation about a quintuple point through 72°.

Hence, $$(KT)^2 = (KL)^4 = (LT)^5 = 1.$$

The abstract generational conditions also require other relations in addition to the above.[*]

THE UNIVERSITY OF CHICAGO, *July* 18, 1900.

---

[*] E. H. Moore, "Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holoedrically Isomorphic with the Symmetric and Alternating Substitution Groups on k Letters" (Proceedings of the London Mathematical Society), vol. XXVIII, No. 597, pp. 357–366. Also, "Concerning Klein's Group of $(n+1)!$ n-ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 341–342, §10, 1900).

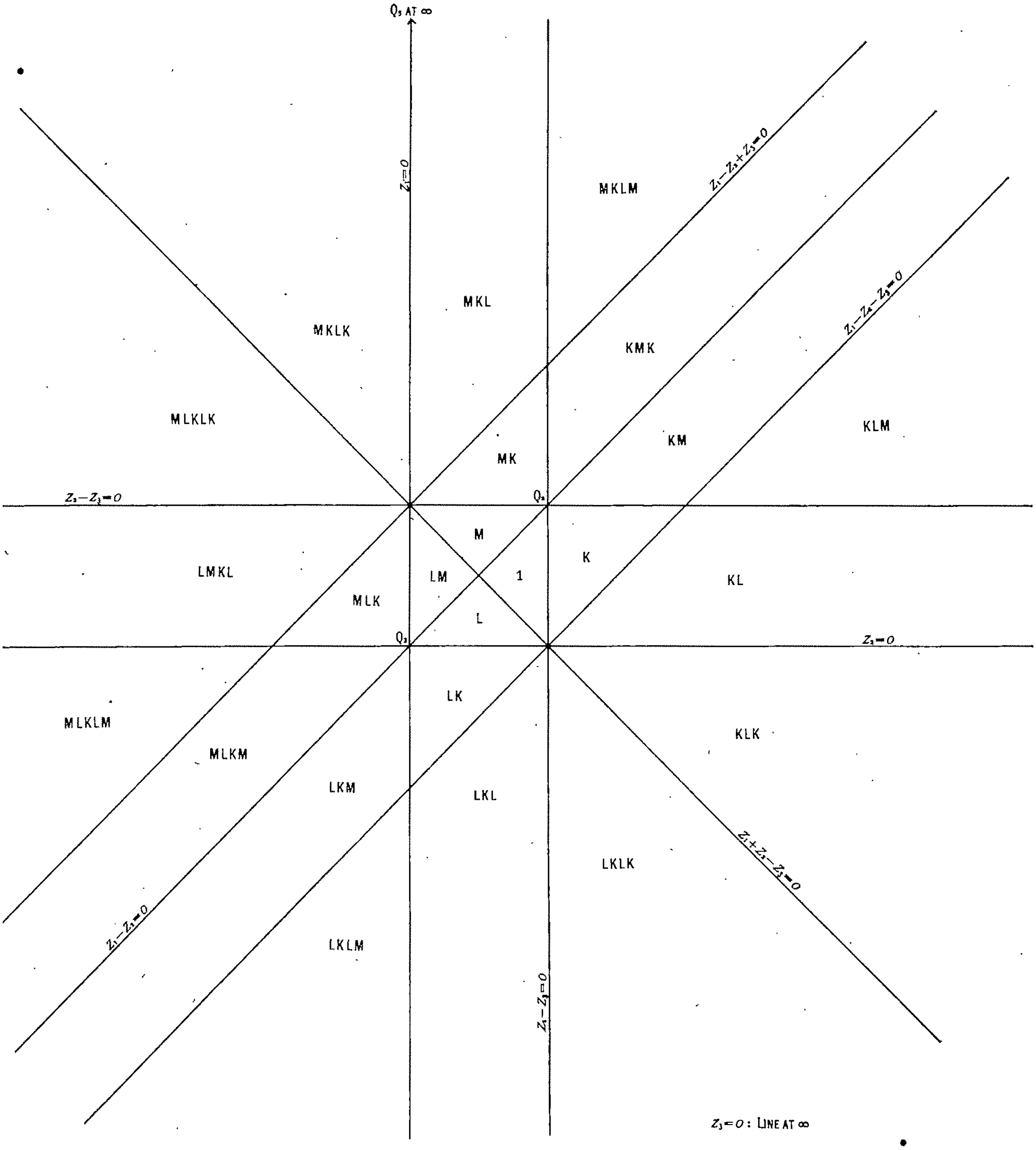

Q_3 AT ∞
$Z_1=0$
MKLM
$Z_1-Z_2+Z_3=0$
MKL
MKLK
KMK
$Z_1-Z_2-Z_3=0$
MLKLK
KLM
KM
MK
$Z_2-Z_3=0$
Q_2
M
K
LMKL
LM
1
KL
MLK
L
Q_1
$Z_2=0$
LK
MLKLM
KLK
MLKM
LKM
LKL
$Z_1+Z_2-Z_3=0$
LKLK
$Z_1-Z_2=0$
LKLM
$Z_1-Z_3=0$
$Z_3=0$: LINE AT ∞

Fig. II.

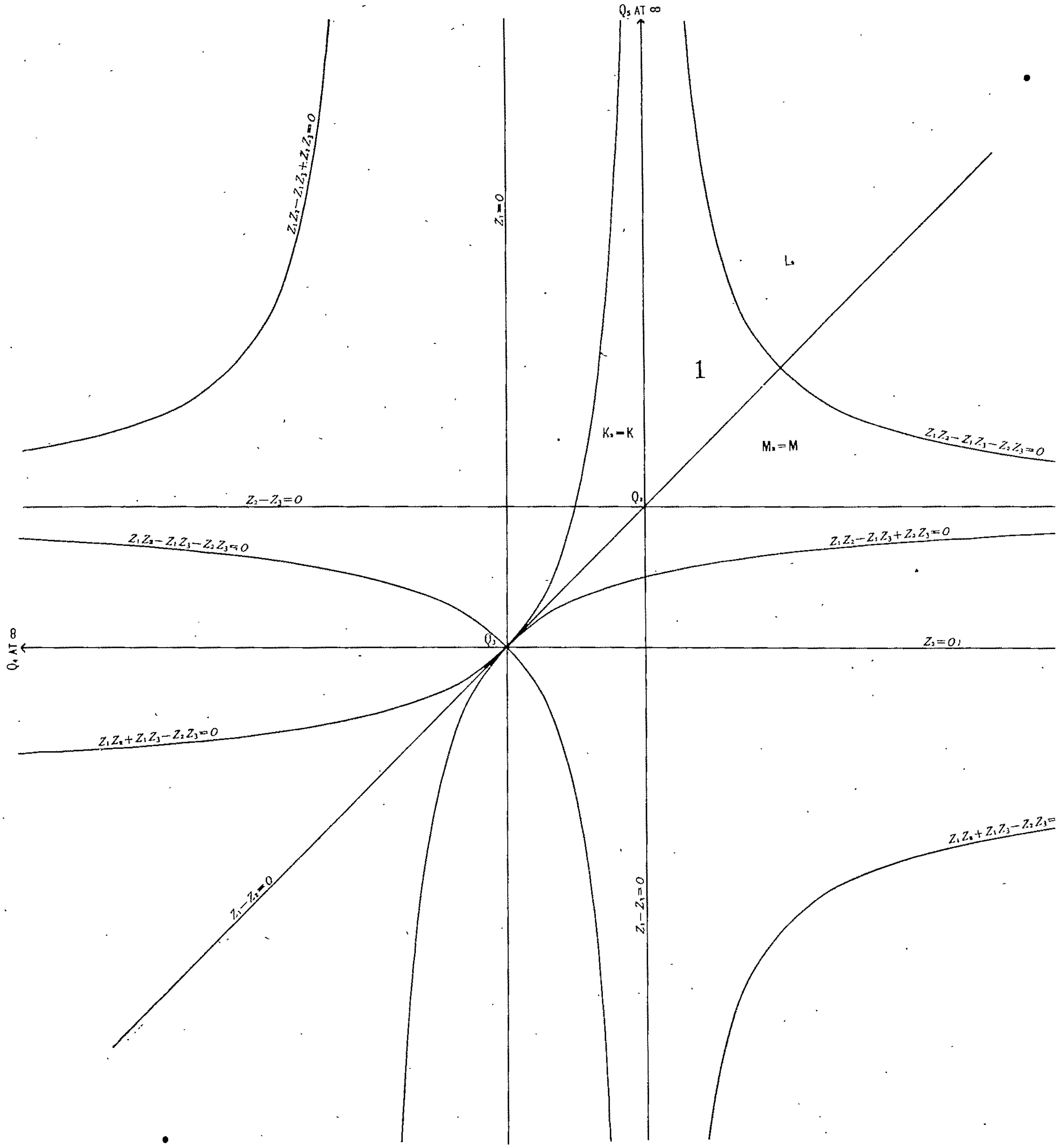

Q_5 AT ∞
$Z_1Z_2-Z_1Z_3+Z_2Z_3=0$
$Z_1=0$
L_2
1
$K_2=K$
$M_2=M$
$Z_1Z_2-Z_1Z_3-Z_2Z_3=0$
Q_2
$Z_2-Z_3=0$
$Z_1Z_2-Z_1Z_3-Z_2Z_3=0$
$Z_1Z_2-Z_1Z_3+Z_2Z_3=0$
Q_4 AT ∞
Q_3
$Z_2=0$
$Z_1Z_2+Z_1Z_3-Z_2Z_3=0$
$Z_1Z_2+Z_1Z_3-Z_2Z_3=$
$Z_1-Z_2=0$
$Z_1-Z_3=0$

Fig. III.

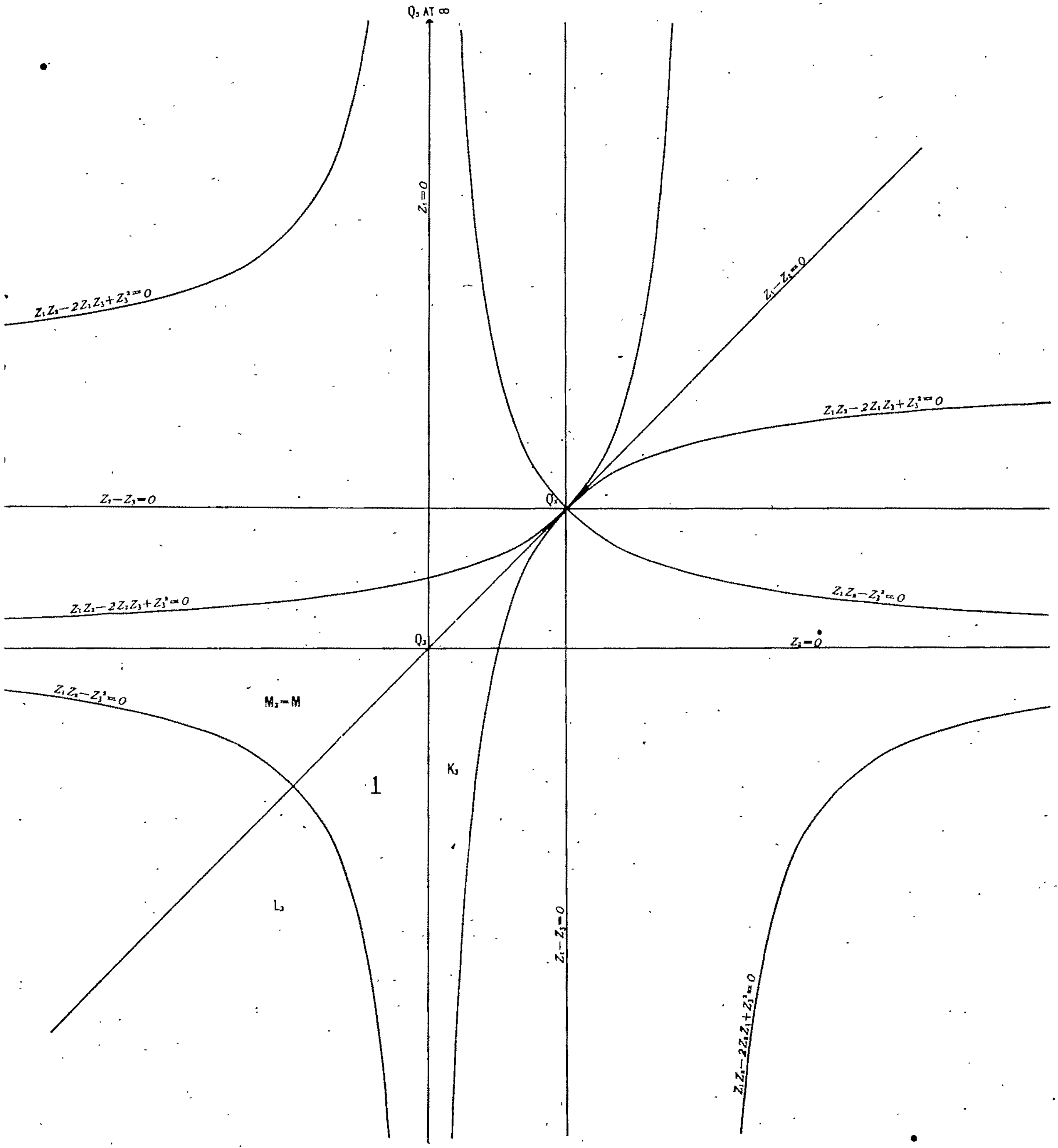

Q_1 AT ∞
$Z_1=0$
$Z_1Z_2-2Z_1Z_3+Z_3^2=0$
$Z_1-Z_2=0$
$Z_1Z_2-2Z_1Z_3+Z_3^2=0$
$Z_2-Z_3=0$
Q_2
$Z_1Z_2-2Z_2Z_3+Z_3^2=0$
$Z_1Z_2-Z_3^2=0$
Q_3
$Z_2=0$
$Z_1Z_2-Z_3^2=0$
$M_2=M$
1
K_3
L_3
$Z_1-Z_3=0$
$Z_1Z_2-2Z_2Z_1+Z_3^2=0$

Fig. IV.

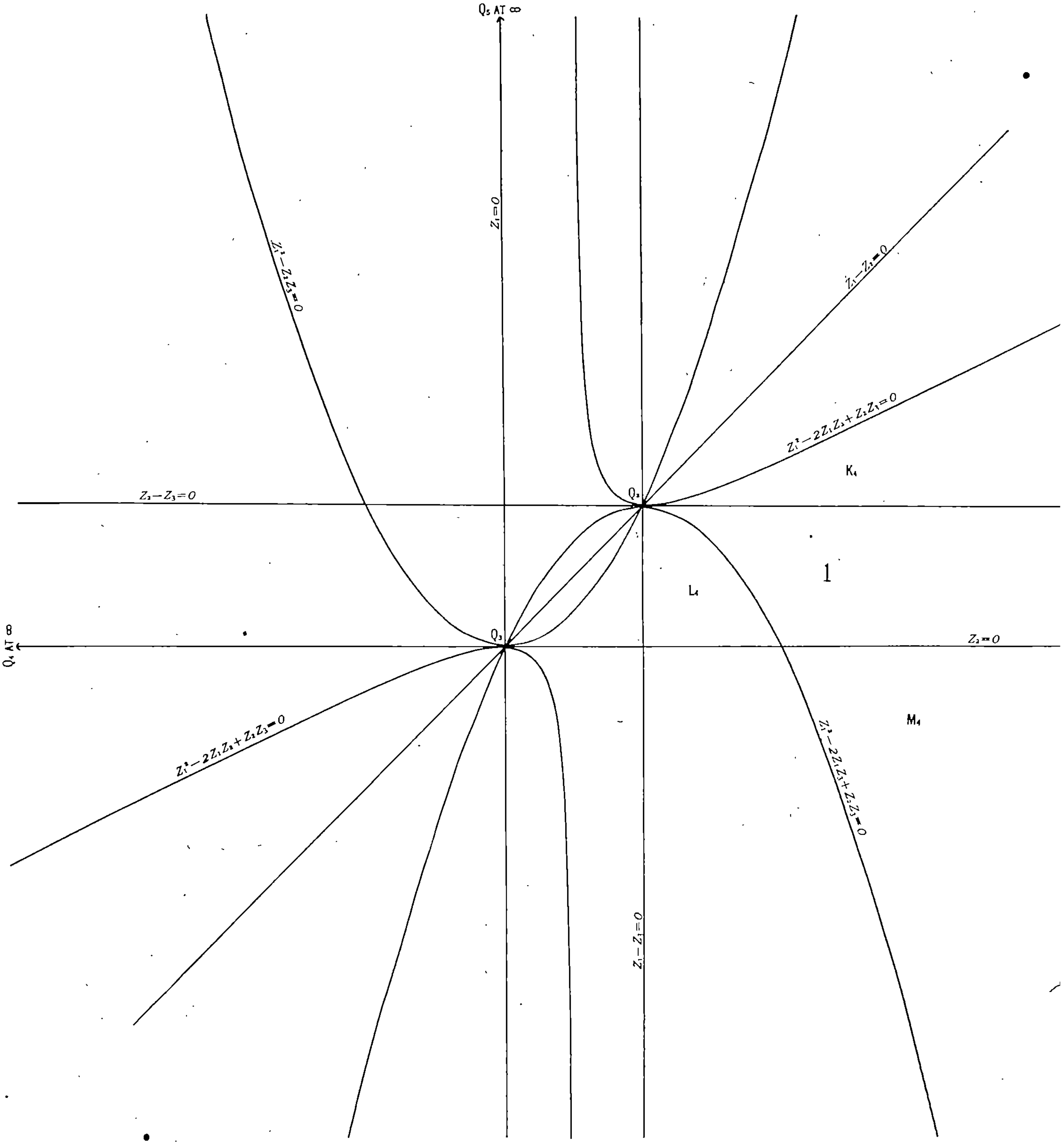

Q_5 AT ∞
$Z_1=0$
$Z_1^2-Z_2Z_3=0$
$Z_1-Z_3=0$
$Z_1^2-2Z_1Z_3+Z_2Z_3=0$
K_4
Q_2
$Z_2-Z_3=0$
1
L_4
Q_3
Q_4 AT ∞
$Z_2=0$
$Z_1^2-2Z_1Z_2+Z_2Z_3=0$
M_4
$Z_1^2-2Z_1Z_3+Z_2Z_3=0$
$Z_1-Z_3=0$

Fig. V.

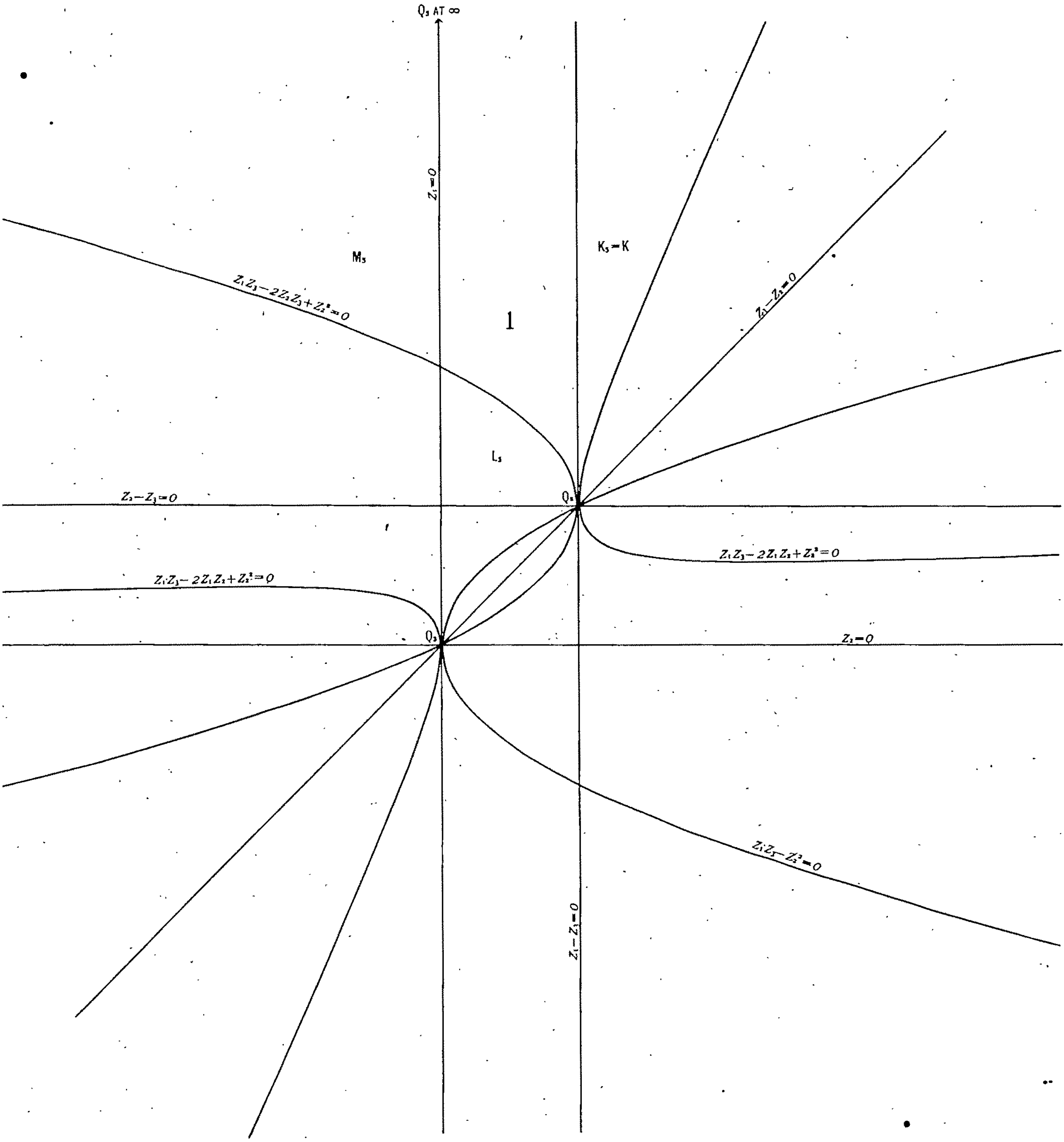

Q_3 AT ∞
$Z_1=0$
M_3
$K_3=K$
1
$Z_1Z_3-2Z_2Z_3+Z_2^2=0$
$Z_1-Z_2=0$
L_3
$Z_2-Z_3=0$
Q_2
$Z_1Z_3-2Z_1Z_2+Z_2^2=0$
$Z_1Z_3-2Z_1Z_2+Z_2^2=0$
Q_1
$Z_2=0$
$Z_1Z_3-Z_2^2=0$
$Z_1-Z_3=0$

Fig. VI.

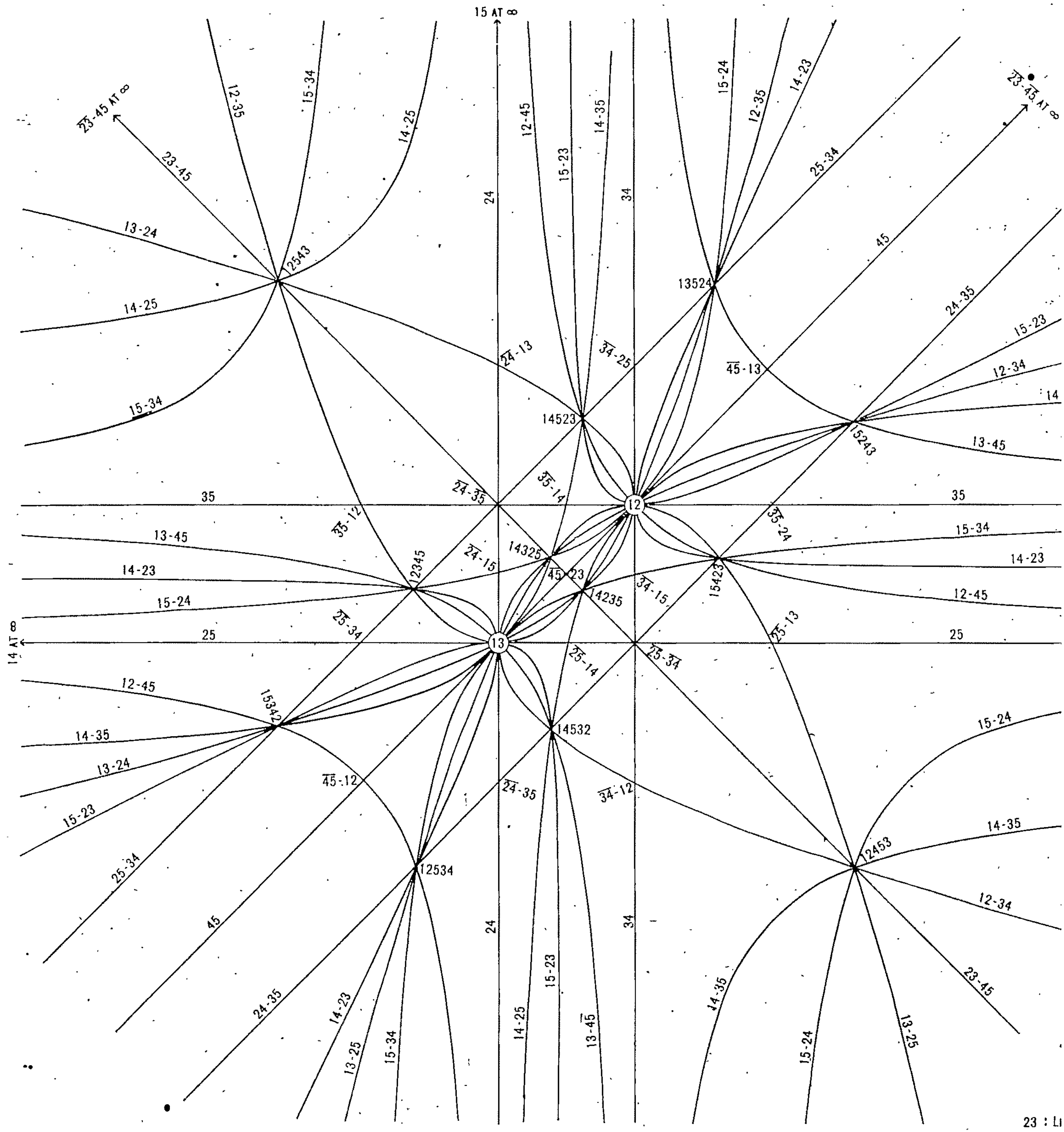
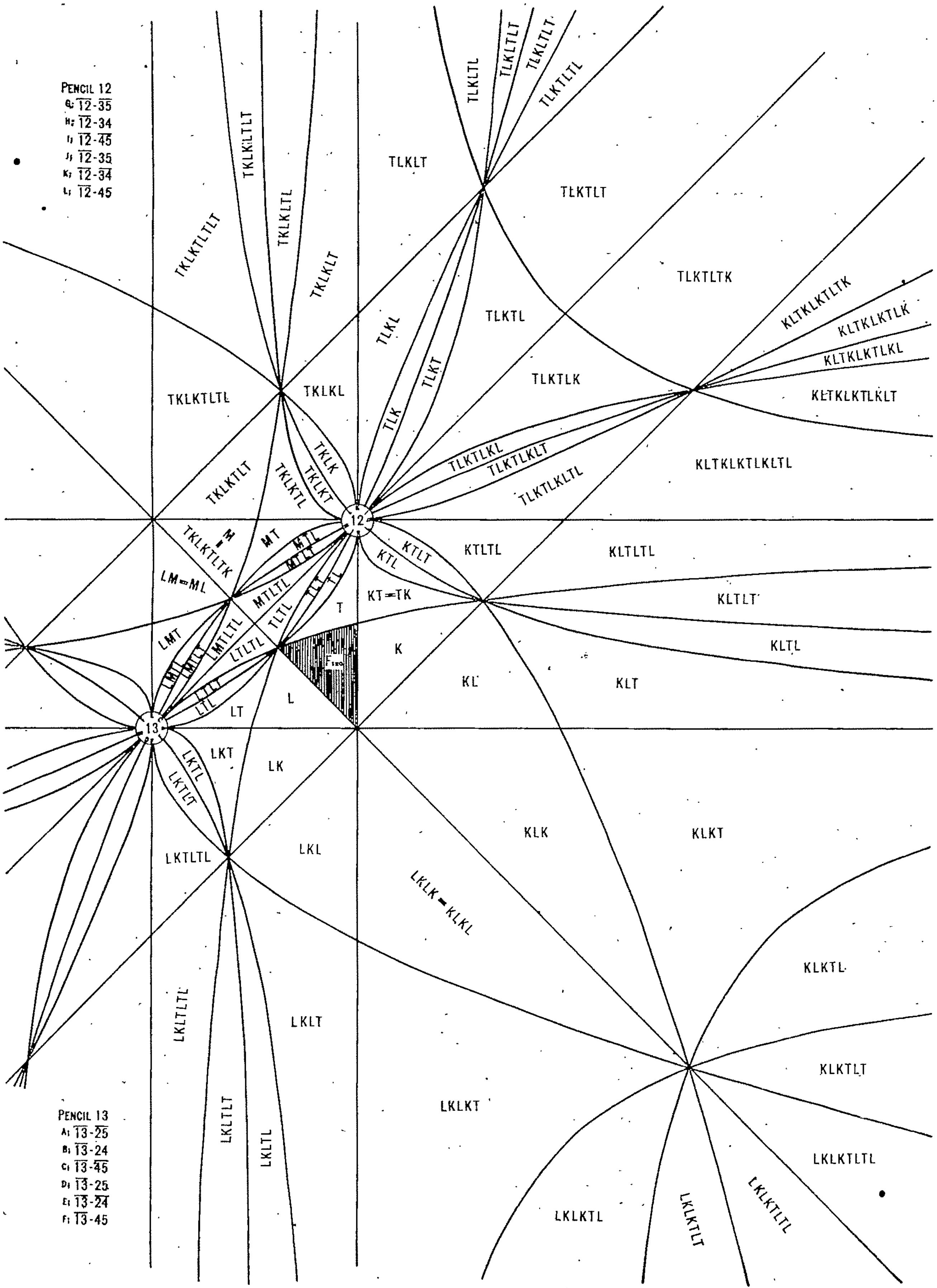

15 AT ∞
$\overline{23}$-45 AT ∞
$\overline{23}$-$\overline{45}$ AT ∞
14 AT ∞
23-45
12-35
15-34
14-25
13-24
2543
14-25
15-34
24
12-45
15-23
14-35
34
15-24
12-35
14-23
25-34
45
13524
24-35
15-23
12-34
14
$\overline{24}$-13
$\overline{34}$-25
$\overline{45}$-13
14523
15243
13-45
$\overline{24}$-$\overline{35}$
$\overline{35}$-14
35
12
35
$\overline{35}$-12
$\overline{35}$-24
13-45
14325
$\overline{24}$-15
15-34
14-23
12345
$\overline{45}$-23
14235
15423
14-23
15-24
$\overline{34}$-15
12-45
$\overline{25}$-34
$\overline{25}$-13
25
13
25
$\overline{25}$-14
$\overline{25}$-$\overline{34}$
12-45
15342
14532
14-35
13-24
$\overline{45}$-12
$\overline{24}$-35
$\overline{34}$-12
15-23
25-34
12534
15-24
14-35
2453
12-34
45
24
34
23-45
24-35
14-23
13-25
15-34
14-25
15-23
13-$\overline{45}$
14-35
15-24
13-25

Fig. X.

23 : Li



PENCIL 12
G: $\overline{12}$-$\overline{35}$
H: $\overline{12}$-34
I: $\overline{12}$-$\overline{45}$
J: $\overline{12}$-35
K: $\overline{12}$-$\overline{34}$
L: $\overline{12}$-45
TKLKTLT
TKLKLTL
TLKLTL
TLKLTLT
TLKLTLT
TLKTLTL
TLKLT
TLKTLT
TKLKTLTLT
TKLKLT
TLKL
TLKTL
TLKTLTK
KLTKLKTLTK
KLTKLKTLK
KLTKLKTLKL
KLTKLKTLKLT
TLKTLK
TKLKTLTL
TKLKL
TLK
TLKT
TLKTLKL
TLKTLKLT
TLKTLKLTL
KLTKLKTLKLTL
TKLK
TKLKT
TKLKTL
TKLKTLT
12
TKLKTLTK
MT
MTL
KTLT
KTL
KTLTL
KLTLTL
LM=ML
MTLTL
TLTL
TL
T
KT=TK
KLTLT
LMT
LMTLTL
LTLTL
F_{120}
K
KLTL
KL
KLT
LTL
LT
L
13
LKT
LKTL
LKTLT
LK
LKTLTL
LKL
KLK
KLKT
LKLK=KLKL
KLKTL
LKLTLTL
LKLT
KLKTLT
LKLTLT
LKLTL
LKLKT
LKLKTLTL
LKLKTL
LKLKTLT
LKLKTLTL
PENCIL 13
A: $\overline{13}$-$\overline{25}$
B: $\overline{13}$-24
C: $\overline{13}$-$\overline{45}$
D: $\overline{13}$-25
E: $\overline{13}$-$\overline{24}$
F: $\overline{13}$-45

Fig. XI.

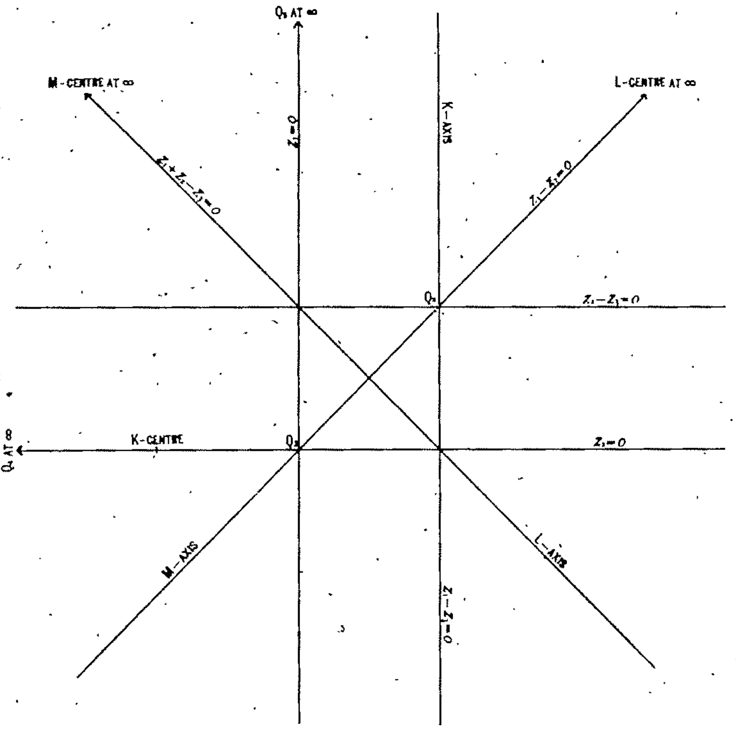


Fig. I.

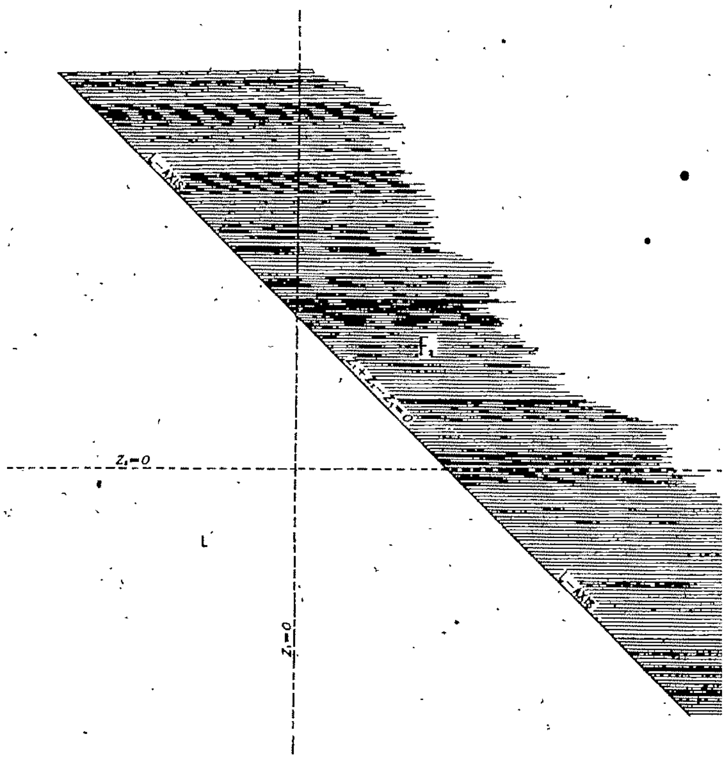


Fig. VII.

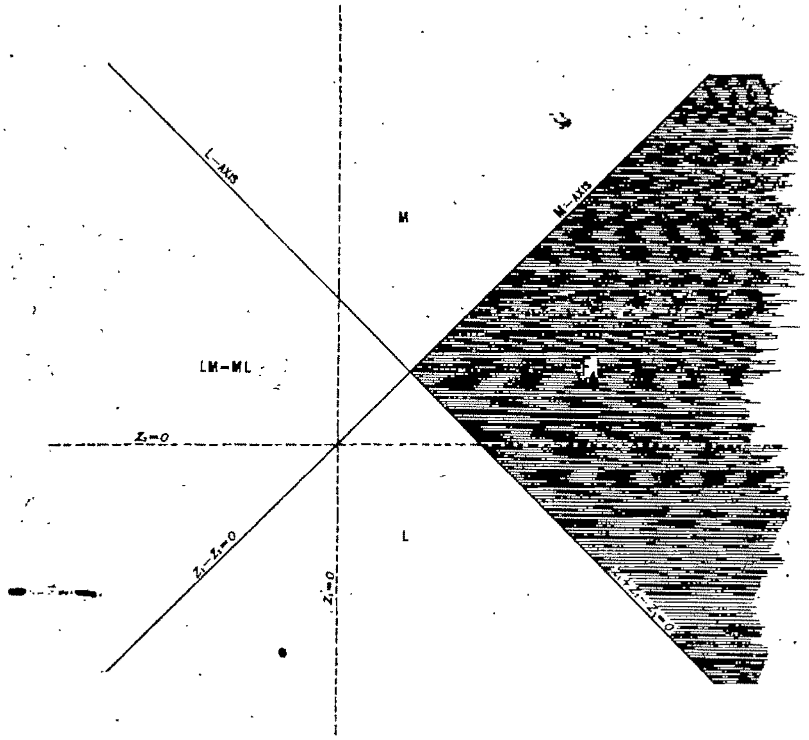


Fig. VIII.

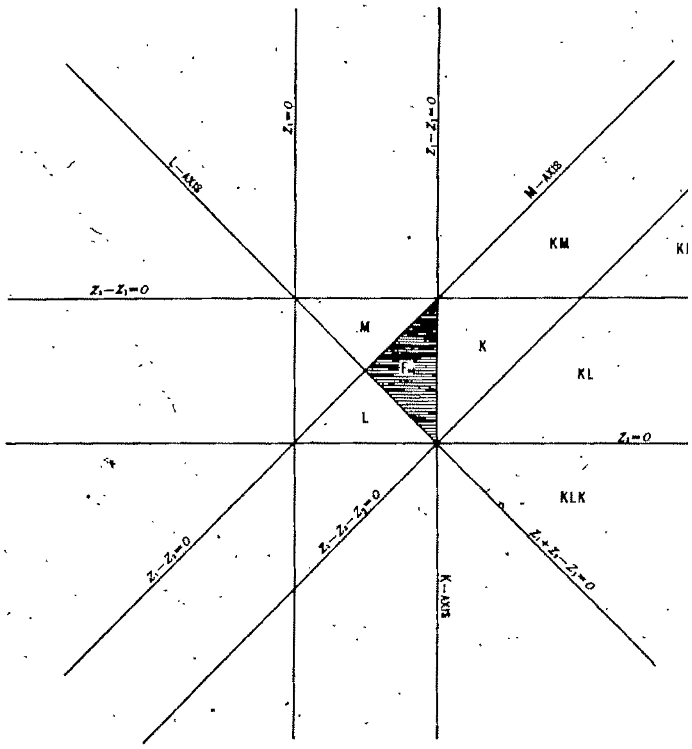


Fig. IX.